Relative Perturbation Theory: (I) Eigenvalue Variations

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Abstract

In this paper, we consider how eigenvalues of a matrix A change when it is perturbed to $\widetilde{A} = D_1^* A D_2$ and how singular values of a (nonsquare) matrix B change when it is perturbed to $\widetilde{B} = D_1^* B D_2$, where D_1 and D_2 are assumed to be close to unitary matrices of suitable dimensions. We have been able to generalize many well-known perturbation theorems, including Hoffman-Wielandt theorem and Weyl-Lidskii theorem. As applications, we obtained bounds for perturbations of graded matrices in both singular value problems and nonnegative definite Hermitian eigenvalue problems.

1 Introduction

Relative perturbation theory for eigensystems and singular systems has been becoming a hot topic in the last five years and ever since It was first studied by Kahan [18] in 1966, later by [1, 6, 8, 9, 29] and most recently by [7, 10, 11, 13, 15, 25].

1.1 What to be Covered?

This paper deals with perturbations of the following kinds:

- **Eigenvalue problems:**
  1. A and $\widetilde{A} = D^* A D$ for Hermitian case, where D is nonsingular and close to I or more generally to a unitary matrix;
  2. A and $\widetilde{A} = D_1^* A D_2$ for general diagonalizable case, where D_1 and D_2 are nonsingular and close to I or more generally to some unitary matrix;

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3. $H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$ for graded nonnegative Hermitian case, where it is assumed that A and $\widetilde{A}$ are nonsingular and often that D is a highly graded diagonal matrix (this assumption is not necessary to our theorems below).

- **Singular value problems:**

1. B and $\widetilde{B} = D_1^* B D_2$, where D_1 and D_2 are nonsingular and close to I or more generally to two unitary matrices;
2. $G = BD$ and $\widetilde{G} = \widetilde{B}D$ for graded case, where it is assumed that B and $\widetilde{B}$ are nonsingular and often that D is a highly graded diagonal matrix (this assumption is not necessary to our theorems below).

The above perturbations include component-wise relative perturbations of the entries in symmetric tridiagonal matrices with zero diagonal [8, 18], in bidiagonal and biacyclic matrices [1, 7, 8], in graded nonnegative Hermitian matrices [9, 25] and in graded matrices of singular value problems [9, 25] and more [10].

1.2 Notation

We will adopt this convention: capital letters denote unperturbed matrices and capital letters with *tilde* denote their perturbed ones. For example, X *is perturbed to* $\widetilde{X}$.

Throughout the paper, capital letters are for matrices, lowercase Latin letters for column vectors or scalars, and lowercase Greek letters for scalars. The following is a detailed list of our notation, but still more notation will be introduced when it appears for the first time.

$\mathbb{C}^{m\times n}$:	the set of $m \times n$ complex matrices;
$\mathbb{C}^m$:	$\mathbb{C}^{m\times 1}$;
$\mathbb{C}$:	$\mathbb{C}^1$;
$\mathbb{R}^{m\times n}$:	the set of $m \times n$ real matrices;
$\mathbb{R}^m$:	$\mathbb{R}^{m\times 1}$;
$\mathbb{R}$:	$\mathbb{R}^1$;
$\mathbb{U}_n$:	the set of $n \times n$ unitary matrices;
$0_{m,n}$:	the $m \times n$ zero matrix (we may simply write 0 instead);
I_n:	the $n \times n$ identity matrix (we may simply write I instead);
X^*:	the complex conjugate of a matrix X;
$\lambda(X)$:	the set of the eigenvalues of X, counted according to their algebraic multiplicities;
$\sigma(X)$:	the set of the singular values of X, counted according to their algebraic multiplicities;

$\sigma_{\min}(X)$:	the smallest singular value of $X \in \mathbb{C}^{n\times n}$;
$\sigma_{\max}(X)$:	the largest singular value of $X \in \mathbb{C}^{m\times n}$;
$\|X\|_2$:	the spectral norm of X, $\sigma_{\max}(X)$;
$\|X\|_F$:	the Frobenius norm of X, $\sqrt{\sum_{i,j} \|x_{ij}\|^2}$, where $X = (x_{ij})$;
$\|X\|_p$:	the p-Hölder operator norm of X to defined later;
$\vert\vert\vert X \vert\vert\vert$:	some unitary invariant norm of X to defined later.

1.3 Organization of the Paper

In §2, we define two kinds of relative distances which will be heavily used in the rest of this paper. It is proved in Appendixes A and B that the relative distances are really (generalized) metrics on the space of nonnegative real numbers or that of nonpositive real numbers and that some of them are actually a metric on $\mathbb{R}$. A brief summary of what we will accomplish in this paper in comparison with well-known perturbation theorems with the metric of absolute value on $\mathbb{C}$ will be conducted in §3. Full statements of these well-known theorems are presented in §3. We devote two sections to present and discuss our theorems. §5 handles nonnegative definite cases, singular value problems and graded cases, while §6 handles the rest of the perturbations listed in §1.1 and singular value problems again for comparison purpose. In §7, we give a brief account of established theorems related to our relative perturbation theorems. We will briefly remark how our relative perturbation theorems can be applied to generalized eigenvalue problems and generalized singular value problems. Finally, our proofs of theorems are presented in §§9—12.

2 Relative Distances

2.1 The p-Relative Distance

Given α, $\beta \in \mathbb{C}$, the *p-relative distance* between them is defined as

$$\mathrm{RelDist}_p(\alpha, \beta) \stackrel{\mathrm{def}}{=} \frac{|\alpha - \beta|}{\sqrt[p]{|\alpha|^p + |\beta|^p}}, \tag{2.1}$$

where $1 \le p \le \infty$. We define, for convenience, $0/0 \stackrel{\mathrm{def}}{=} 0$. $\mathrm{RelDist}_\infty$ was first used by Deift, Demmel, Li, and Tomei [6] for defining relative gaps.

Proposition 2.1 *Let $1 \le p \le \infty$ and α, $\beta \in \mathbb{C}$.*

1. $\mathrm{RelDist}_p(\alpha, \beta) \ge 0$ *and the equality sign holds if and only if* $\alpha = \beta$*;*
2. $\mathrm{RelDist}_p(\alpha, \beta) = \mathrm{RelDist}_p(\beta, \alpha)$*;*
3. $\mathrm{RelDist}_p(\xi\alpha, \xi\beta) = \mathrm{RelDist}_p(\alpha, \beta)$ *for all* $0 \ne \xi \in \mathbb{C}$*;*
4. $\mathrm{RelDist}_p(1/\alpha, 1/\beta) = \mathrm{RelDist}_p(\alpha, \beta)$ *for* $\alpha \ne 0$ *and* $\beta \ne 0$*;*
5. $\mathrm{RelDist}_p(\alpha, \beta) \le 2^{1-1/p}$ *and the equality sign holds if and only if* $\alpha = -\beta \ne 0$*;*
6. $\mathrm{RelDist}_p(\alpha, 0) \equiv 1$ *if* $\alpha \ne 0$*;* $\mathrm{RelDist}_p(\alpha, \beta) > 1$ *for* $p > 1$ *and* $\mathrm{RelDist}_1(\alpha, \beta) = 1$*, if* $\alpha\beta < 0$*; Finally,* $\mathrm{RelDist}_p(\alpha, \beta) < 1$ *for all* p *if* $\alpha\beta > 0$*.*
7. $\mathrm{RelDist}_p(\alpha, \beta)$ *increases as* p *does.*
8. *if* α, α_1, β, $\beta_1 \in \mathbb{R}$ *and* $\alpha \le \alpha_1 \le \beta_1 \le \beta$ *and* $\alpha_1\beta_1 \ge 0$*, then*

$$\mathrm{RelDist}_p(\alpha, \beta) \ge \mathrm{RelDist}_p(\alpha_1, \beta_1). \tag{2.2}$$

Moreover if either $\alpha < \alpha_1$ *or* $\beta_1 < \beta$ *holds, the inequality (2.2) is strict.*

Proof: Properties 1–6 are trivial. Property 7 holds because $\sqrt[p]{|\alpha|^p + |\beta|^p}$ is a decreasing function of p for $1 \le p \le \infty$. To prove Property 8, it suffices to show that

$$\mathrm{RelDist}_p(\alpha, \beta) > \mathrm{RelDist}_p(\alpha, \beta_1), \tag{2.3}$$

where $\alpha \le \beta_1 < \beta$ and $\alpha\beta_1 \ge 0$. Consider function $f(\xi)$ defined by

$$f(\xi) \stackrel{\mathrm{def}}{=} \frac{1 - \xi}{\sqrt[p]{1 + |\xi|^p}}, \quad \text{where } -1 \le \xi \le 1.$$

We claim that the function $f(\xi)$ so defined is strictly monotonically decreasing. This is true if $p = \infty$. When $p < \infty$, set $h(\xi) \stackrel{\mathrm{def}}{=} [f(\xi)]^p$. Because for $0 < \xi < 1$

$$h'(\xi) = -\frac{p(1 - \xi)^{p-1}(1 + \xi^{p-1})}{(1 + \xi^p)^2} < 0,$$

$f(\xi)$ is strictly monotonically decreasing for $0 \le \xi \le 1$. For $-1 \le \xi \le 0$, set $g(\xi) \stackrel{\text{def}}{=} h(-\xi)$. Since for $0 < \xi < 1$

$$g'(\xi) = \frac{p(1+\xi)^{p-1}(1-\xi^{p-1})}{(1+\xi^p)^2} > 0,$$

$g(\xi)$ is strictly monotonically increasing for $0 \le \xi \le 1$, and thus $h(\xi)$ and $f(\xi)$ is strictly monotonically decreasing for $-1 \le \xi \le 0$. This completes the proof of that the function $f(\xi)$ is strictly monotonically decreasing. There are several cases to deal with in order to prove (2.3).

1. if $\alpha \ge 0$, then $0 \le \alpha/\beta < \alpha/\beta_1 \le 1$ and

$$\text{RelDist}_p(\alpha,\beta) = f(\alpha/\beta) > f(\alpha/\beta_1) = \text{RelDist}_p(\alpha,\beta_1);$$

2. if $\beta \le 0$, then $0 \le \beta/\alpha < \beta_1/\alpha \le 1$ and

$$\text{RelDist}_p(\alpha,\beta) = f(\beta/\alpha) > f(\beta_1/\alpha) = \text{RelDist}_p(\alpha,\beta_1);$$

3. if $\beta_1 \le 0 < \beta$, then $0 \le \beta_1/\alpha \le 1$. Let ξ_0 be the one of α/β and β/α which lies in $[-1, 0]$. Now if $\alpha = \beta_1 = 0$, (2.3) is trivial; otherwise either $\alpha \le \beta_1 < 0 < \beta$ or $\alpha < \beta_1 = 0 < \beta$ is true, and thus $-1 \le \xi_0 < 0 \le \beta_1/\alpha \le 1$, so we have

$$\text{RelDist}_p(\alpha,\beta) = f(\xi_0) > f(\beta_1/\alpha) = \text{RelDist}_p(\alpha,\beta_1),$$

as was to be shown.

The proof of Property 8 is completed. ∎

Remark: In Property 8 of Proposition 2.1, the assumption $\alpha_1\beta_1 \ge 0$ is essential. This can be seen by noting that for $\beta > \alpha > 0$, $-\alpha \le -\alpha < \alpha < \beta$ while

$$\text{RelDist}_p(-\alpha,\beta) = \frac{\alpha+\beta}{\sqrt[p]{\alpha^p+\beta^p}} < 2^{1-1/p} = \text{RelDist}_p(-\alpha,\alpha).$$

Now, we introduce another global notation of this paper. Henceforth p and q are reserved for a *dual number pair* as defined below

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \text{where } 1 \le p \le \infty \text{ and } 1 \le q \le \infty.$$

In general, when people say the relative perturbation in a real number α is at most ϵ, it is meant that α is perturbed to another real number β in the sense that if we write $\beta = \alpha(1+\delta)$ then $\delta \in \mathbb{R}$ and $|\delta| \le \epsilon$ (see, e.g., [8]), which is also equivalently to say

$$\left|\frac{\beta}{\alpha} - 1\right| \le \epsilon.$$

So it would be interesting to relate our p-relative distance to this common sense of relative perturbations.

Proposition 2.2 *Let $0 \le \epsilon < 1$, and $\alpha, \beta \in \mathbb{R}$. We have the following:*

$$\left|\frac{\beta}{\alpha} - 1\right| \le \epsilon \Rightarrow \mathrm{RelDist}_p(\alpha, \beta) \le \epsilon, \tag{2.4}$$

and

$$\mathrm{RelDist}_1(\alpha,\beta) \le \epsilon \quad \Rightarrow \quad \max\left\{\left|\frac{\beta}{\alpha}-1\right|, \left|\frac{\alpha}{\beta}-1\right|\right\} \le \frac{2\epsilon}{1-\epsilon}; \tag{2.5}$$

$$\mathrm{RelDist}_2(\alpha,\beta) \le \epsilon \quad \Rightarrow \quad \max\left\{\left|\frac{\beta}{\alpha}-1\right|, \left|\frac{\alpha}{\beta}-1\right|\right\} \le \frac{\sqrt{2}\,\epsilon}{1-\epsilon}; \tag{2.6}$$

$$\mathrm{RelDist}_\infty(\alpha,\beta) \le \epsilon \quad \Rightarrow \quad \max\left\{\left|\frac{\beta}{\alpha}-1\right|, \left|\frac{\alpha}{\beta}-1\right|\right\} \le \frac{\epsilon}{1-\epsilon}. \tag{2.7}$$

For general $1 \le p \le \infty$, if $2^{1/p}\epsilon < 1$ we have

$$\mathrm{RelDist}_p(\alpha,\beta) \le \epsilon \Rightarrow \max\left\{\left|\frac{\beta}{\alpha}-1\right|, \left|\frac{\alpha}{\beta}-1\right|\right\} \le \frac{2^{1/p}\,\epsilon}{1-2^{1/p}\epsilon}. \tag{2.8}$$

Asymptotically,

$$\lim_{\beta\to\alpha} \frac{\mathrm{RelDist}_p(\alpha,\beta)}{\left|\frac{\beta}{\alpha}-1\right|} = 2^{1/p}, \tag{2.9}$$

thus (2.4), (2.5), (2.6) and (2.7), (2.8) are at least asymptotically sharp.

Proof: (2.4) is trivial to show since $\beta - \alpha = \alpha(1+\delta) - \alpha = \alpha\delta$. To prove (2.5), (2.6) and (2.7), we set either $\xi = \beta/\alpha$ or $\xi = \alpha/\beta$. Then $\xi > 0$. It follows from the left-hand side of (2.5) that

$$\frac{|\xi - 1|}{\xi + 1} \le \epsilon \Rightarrow |\xi - 1| \le \epsilon(\xi+1) = \epsilon(\xi-1) + 2\epsilon.$$

So if $\xi \ge 1$, one deduces $\xi - 1 \le \frac{2\epsilon}{1-\epsilon}$; and if $\xi \le 1$ one has $1 - \xi \le \frac{2\epsilon}{1+\epsilon}$. This completes the proof of (2.5). The proof of (2.7) is analogous. So is that of (2.8) by noting that $2^{1/p}\mathrm{RelDist}_p(\alpha,\beta) \ge \mathrm{RelDist}_\infty(\alpha,\beta)$. To show (2.6), we see that the left-hand side of (2.6) implies $\frac{|\xi-1|}{\sqrt{1+\xi^2}} \stackrel{\mathrm{def}}{=} \eta \le \epsilon$. So

$$(\xi-1)^2 = \eta^2(\xi^2+1) \Rightarrow \xi^2 - \frac{2}{1-\eta^2}\xi + 1 = 0$$

solving which gives

$$\xi = \frac{1 \pm \eta\sqrt{2-\eta^2}}{1-\eta^2} \Rightarrow \xi - 1 = \frac{\pm\eta\sqrt{2-\eta^2} + \eta^2}{1-\eta^2}.$$

Hence

$$|\xi - 1| \le \frac{\eta\sqrt{2-\eta^2}+\eta^2}{1-\eta^2} = \frac{\eta}{1-\eta}\cdot\frac{\sqrt{2-\eta^2}+\eta}{1+\eta} \le \frac{\epsilon}{1-\epsilon}\cdot\sqrt{2}$$

since $\frac{\sqrt{2-\eta^2}+\eta}{1+\eta}$ is decreasing for $0 \le \eta \le 1$. ∎

Proposition 2.3 *Let* $\widetilde{\alpha} = \alpha(1+\delta_1)$ *and* $\widetilde{\beta} = \beta(1+\delta_2)$. *If* $|\delta_i| \le \epsilon < 1$, *then*

$$\frac{\mathrm{RelDist}_p(\alpha,\beta)}{1-\epsilon} + \frac{\epsilon}{1-\epsilon} \ge \mathrm{RelDist}_p(\alpha,\widetilde{\beta}) \ge \frac{\mathrm{RelDist}_p(\alpha,\beta)}{1+\epsilon} - \frac{\epsilon}{1+\epsilon}, \quad (2.10)$$

$$\frac{\mathrm{RelDist}_p(\alpha,\beta)}{1-\epsilon} + \frac{2^{1/q}\epsilon}{1-\epsilon} \ge \mathrm{RelDist}_p(\widetilde{\alpha},\widetilde{\beta}) \ge \frac{\mathrm{RelDist}_p(\alpha,\beta)}{1+\epsilon} - \frac{2^{1/q}\epsilon}{1+\epsilon}. \quad (2.11)$$

Proof: We will only provide a proof of (2.11). Since $|\alpha|(1-\epsilon) \le |\widetilde{\alpha}| \le |\alpha|(1+\epsilon)$ and $|\beta|(1-\epsilon) \le |\widetilde{\beta}| \le |\beta|(1+\epsilon)$,

$$\begin{aligned}
\mathrm{RelDist}_p(\widetilde{\alpha},\widetilde{\beta}) &= \frac{|\widetilde{\alpha}-\widetilde{\beta}|}{\sqrt[p]{|\widetilde{\alpha}|^p + |\widetilde{\beta}|^p}} \\
&\ge \frac{|\alpha-\beta| - |\alpha\delta_1 - \beta\delta_2|}{\sqrt[p]{|\alpha|^p+|\beta|^p}(1+\epsilon)} \\
&\ge \frac{|\alpha-\beta| - \sqrt[p]{|\alpha|^p+|\beta|^p}\sqrt[q]{\epsilon^q+\epsilon^q}}{\sqrt[p]{|\alpha|^p+|\beta|^p}(1+\epsilon)} \\
&= \frac{\mathrm{RelDist}_p(\alpha,\beta)}{1+\epsilon} - \frac{2^{1/q}\epsilon}{1+\epsilon}, \\
\mathrm{RelDist}_p(\widetilde{\alpha},\widetilde{\beta}) &\le \frac{|\alpha-\beta| + |\alpha\delta_1 - \beta\delta_2|}{\sqrt[p]{|\alpha|^p+|\beta|^p}(1-\epsilon)} \\
&\le \frac{|\alpha-\beta| + \sqrt[p]{|\alpha|^p+|\beta|^p}\sqrt[q]{\epsilon^q+\epsilon^q}}{\sqrt[p]{|\alpha|^p+|\beta|^p}(1-\epsilon)} \\
&= \frac{\mathrm{RelDist}_p(\alpha,\beta)}{1-\epsilon} + \frac{2^{1/q}\epsilon}{1-\epsilon},
\end{aligned}$$

as were to be shown. ∎

Proposition 2.4 below shows how to bound $\mathrm{RelDist}_p(\alpha^2,\beta^2)$ by $\mathrm{RelDist}_p(\alpha,\beta)$, and vice versa.

Proposition 2.4 *Let* $\alpha, \beta \in \mathbb{C}$. *For* $1 \le p \le \infty$,

$$\mathrm{RelDist}_p(\alpha^2,\beta^2) \le 2\,\mathrm{RelDist}_p(\alpha,\beta). \quad (2.12)$$

If, moreover, $\alpha, \beta \in \mathbb{R}$ *and* $\alpha\beta \ge 0$, *then*

$$\mathrm{RelDist}_p(\alpha,\beta) \le \mathrm{RelDist}_p(\alpha^2,\beta^2). \quad (2.13)$$

Proof: There is nothing to prove if $\alpha = \beta = 0$. Assume at least one of the two is not zero.

$$\begin{aligned}
\mathrm{RelDist}_p(\alpha^2, \beta^2) &= \frac{|\alpha^2 - \beta^2|}{(|\alpha|^{2p} + |\beta|^{2p})^{1/p}} \\
&= \frac{|\alpha + \beta| \times (|\alpha|^p + |\beta|^p)^{1/p}}{(|\alpha|^{2p} + |\beta|^{2p})^{1/p}} \times \frac{|\alpha - \beta|}{(|\alpha|^p + |\beta|^p)^{1/p}} \\
&\leq \frac{2^{1-1/2p}(|\alpha|^{2p} + |\beta|^{2p})^{1/2p} \times 2^{1/2p}(|\alpha|^{2p} + |\beta|^{2p})^{1/2p}}{(|\alpha|^{2p} + |\beta|^{2p})^{1/p}} \mathrm{RelDist}_p(\alpha, \beta) \\
&= 2\,\mathrm{RelDist}_p(\alpha, \beta)
\end{aligned}$$

which proves (2.12). To prove (2.13), without loss of any generality, we may assume $\alpha, \beta \geq 0$. Notice that $\alpha + \beta \geq (\alpha^{2p} + \beta^{2p})^{1/2p}$ and $(\alpha^p + \beta^p)^{1/p} \geq (\alpha^{2p} + \beta^{2p})^{1/2p}$. So

$$\begin{aligned}
\mathrm{RelDist}_p(\alpha, \beta) &= \frac{|\alpha^2 - \beta^2|}{(|\alpha|^{2p} + |\beta|^{2p})^{1/p}} \frac{(|\alpha|^{2p} + |\beta|^{2p})^{1/p}}{(\alpha + \beta)(|\alpha|^p + |\beta|^p)^{1/p}} \\
&\leq \mathrm{RelDist}_p(\alpha^2, \beta^2),
\end{aligned}$$

as was to be shown. ∎

Let $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\widetilde{\alpha}_1, \cdots, \widetilde{\alpha}_n\}$ be two sequences of n *real* numbers in ascending (descending) order respectively, i.e.,

$$\alpha_1 \leq \cdots \leq \alpha_n, \quad \widetilde{\alpha}_1 \leq \cdots \leq \widetilde{\alpha}_n, (\text{or } \alpha_1 \geq \cdots \geq \alpha_n, \quad \widetilde{\alpha}_1 \geq \cdots \geq \widetilde{\alpha}_n). \tag{2.14}$$

Now we consider some partial solutions to the question: *What are the best one-one pairings between the α_i's and the $\widetilde{\alpha}_j$'s under certain measures?.*

Proposition 2.5 *If all α_i's and $\widetilde{\alpha}_j$'s are nonnegative, then*

$$\max_{1 \leq i \leq n} \mathrm{RelDist}_p(\alpha_i, \widetilde{\alpha}_i) = \min_{\tau} \max_{1 \leq i \leq n} \mathrm{RelDist}_p(\alpha_i, \widetilde{\alpha}_{\tau(i)}),$$

where the minimization is taken over all permutations τ of $\{1, 2, \cdots, n\}$.

Proof: For any permutation τ of $\{1, 2, \cdots, n\}$, the idea of our proof is to construct $n+1$ permutations τ_j such that

$$\tau_0 = \tau, \quad \tau_n = \text{identity permutation}$$

and for $j = 0, 1, 2, \cdots, n-1$

$$\max_{1 \leq i \leq n} \mathrm{RelDist}_p(\alpha_i, \widetilde{\alpha}_{\tau_j(i)}) \geq \max_{1 \leq i \leq n} \mathrm{RelDist}_p(\alpha_i, \widetilde{\alpha}_{\tau_{j+1}(i)}).$$

The construction of these τ_j's goes as follows: Set $\tau_0 = \tau$. Given τ_j, if $\tau_j(j+1) = j+1$, set $\tau_{j+1} = \tau_j$; otherwise define

$$\tau_{j+1}(i) = \begin{cases} \tau_j(i), & \text{if } \tau_j^{-1}(j+1) \neq i \neq j+1, \\ j+1, & \text{if } i = j+1, \\ \tau_j(j+1), & \text{if } i = \tau_j^{-1}(j+1). \end{cases}$$

With Property 8 in Proposition 2.1, it is easy to prove by induction that such constructed τ_j's have the desired properties. ∎

Remark. Proposition 2.5 may fail if not all of the α_i's and $\widetilde{\alpha}_j$'s are of the same sign. A *counterexample* is as follows: $n = 2$ and

$$\alpha_1 = -2 < \alpha_2 = 1 \quad \text{and} \quad \widetilde{\alpha}_1 = 2 < \widetilde{\alpha}_2 = 4.$$

Another point we want to make is that given two sequences of α_i's and $\widetilde{\alpha}_j$'s as above, generally we do not have

$$\sum_{i=1}^{n} [\mathrm{RelDist}_2(\alpha_i, \widetilde{\alpha}_i)]^2 = \min_{\tau} \sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\alpha_i, \widetilde{\alpha}_{\tau(i)})\right]^2. \tag{2.15}$$

(2.15) may even fail when all α_i, $\widetilde{\alpha}_j > 0$. Here is a *counterexample*: $n = 2$

$$0 < \alpha_1 < \widetilde{\alpha}_1 < \alpha_2 = \widetilde{\alpha}_2/2 < \widetilde{\alpha}_2,$$

where α_1 is sufficiently close to 0, and $\widetilde{\alpha}_1$ is sufficiently close to α_2 which is fixed. Since as $\alpha_1 \to 0^+$ and $\widetilde{\alpha}_1 \to \alpha_2^-$

$$\begin{aligned} [\mathrm{RelDist}_2(\alpha_1, \widetilde{\alpha}_2)]^2 + [\mathrm{RelDist}_2(\alpha_2, \widetilde{\alpha}_1)]^2 &\to 1, \\ [\mathrm{RelDist}_2(\alpha_1, \widetilde{\alpha}_1)]^2 + [\mathrm{RelDist}_2(\alpha_2, \widetilde{\alpha}_2)]^2 &\to 1 + \frac{1}{\sqrt{5}}, \end{aligned}$$

(2.15) must fail for some $0 < \alpha_1 < \widetilde{\alpha}_1 < \alpha_2 = \widetilde{\alpha}_2/2 < \widetilde{\alpha}_2$. But we still have Proposition 2.6 below.

Proposition 2.6 *Let α_i's and $\widetilde{\alpha}_j$'s be as described above and in ascending order. Assume that both sequences contain exactly k negative numbers and $n-k$ positive numbers, i.e.,*

$$\alpha_1 \leq \cdots \alpha_k < 0 < \alpha_{k+1} \leq \cdots \alpha_n, \quad \textit{and} \quad \widetilde{\alpha}_1 \leq \cdots \widetilde{\alpha}_k < 0 < \widetilde{\alpha}_{k+1} \leq \cdots \widetilde{\alpha}_n.$$

Then given a permutation σ of $\{1, 2, \cdots, n\}$, there exists another permutation τ of $\{1, 2, \cdots, n\}$ such that

$$1 \leq \tau(j) \leq k \quad \textit{for } 1 \leq j \leq k$$

and

$$\sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\alpha_i, \widetilde{\alpha}_{\sigma(i)})\right]^2 \geq \sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\alpha_i, \widetilde{\alpha}_{\tau(i)})\right]^2.$$

The proof of this proposition depends heavily on Property 6 of Proposition 2.1.

Let k be an positive integers, and set

$$\alpha_{n+1} = \cdots = \alpha_{n+k} = \widetilde{\alpha}_{n+1} = \cdots = \widetilde{\alpha}_{n+k} = 0.$$

Appending these 0's to the two previous sequences, we have two larger sequences, each of which has at least k zeros. The following proposition says that it is always better to pair zeros with zeros.

Proposition 2.7 *Given a permutation σ of $\{1, 2, \cdots, n+k\}$, there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\sum_{i=1}^{n+k} \left[\mathrm{RelDist}_2(\alpha_i, \widetilde{\alpha}_{\sigma(i)})\right]^2 \geq \sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\alpha_i, \widetilde{\alpha}_{\tau(i)})\right]^2.$$

A combination of Propositions 2.6 and 2.7 illustrates two things:

1. It is always better to pair zeros to zeros as many as possible;
2. It is always better to pair numbers to these of the same signs as many as possible.

2.2 Barlow-Demmel-Veselić Relative Distance

We introduce another notion of *relative distance:* $\widetilde{\mathrm{RelDist}}$ which is defined as follows.

$$\widetilde{\mathrm{RelDist}}(\alpha, \beta) \stackrel{\mathrm{def}}{=} \frac{|\alpha - \beta|}{\sqrt{|\alpha\beta|}}. \tag{2.16}$$

We treat $0/0 \equiv 0$ and $1/0 = \infty$. We call $\widetilde{\mathrm{RelDist}}(\alpha, \beta)$ the *Barlow-Demmel-Veselić Relative Distance* between α and β because it was first used by Barlow and Demmel [1] and Demmel and Veselić [9] for defining relative gaps between the spectra of two matrices. Regarding to $\widetilde{\mathrm{RelDist}}$, we have

Proposition 2.8 *Let α, $\beta \in \mathbb{C}$.*

1. *$\widetilde{\mathrm{RelDist}}(\alpha, \beta) \geq 0$ and the equality sign holds if and only if $\alpha = \beta$;*
2. *$\widetilde{\mathrm{RelDist}}(\alpha, \beta) = \widetilde{\mathrm{RelDist}}(\beta, \alpha)$;*
3. *$\widetilde{\mathrm{RelDist}}(\xi\alpha, \xi\beta) = \widetilde{\mathrm{RelDist}}(\alpha, \beta)$ for all $0 \neq \xi \in \mathbb{C}$;*
4. *$\widetilde{\mathrm{RelDist}}(1/\alpha, 1/\beta) = \widetilde{\mathrm{RelDist}}(\alpha, \beta)$ for $\alpha \neq 0$ and $\beta \neq 0$;*
5. *$\widetilde{\mathrm{RelDist}}(\alpha, 0) = \infty$ if $\alpha \neq 0$;*

6. if $\alpha, \alpha_1, \beta, \beta_1 \in \mathbb{R}$ *and* $\alpha \le \alpha_1 \le \beta_1 \le \beta$ *and* $\alpha\beta \ge 0$*, then*

$$\widetilde{\mathrm{RelDist}}(\alpha,\beta) \ge \widetilde{\mathrm{RelDist}}(\alpha_1,\beta_1). \tag{2.17}$$

Proof: Properties 1–5 are trivial. To prove Property 6, it suffices to show that

$$\widetilde{\mathrm{RelDist}}(\alpha,\beta) \ge \widetilde{\mathrm{RelDist}}(\alpha,\beta_1), \tag{2.18}$$

where $0 \le \alpha \le \beta_1 < \beta$. Since the function $\frac{1}{\xi} - \xi$ for $0 \le \xi \le 1$ is monotonically decreasing and $0 \le \alpha/\beta \le \alpha/\beta_1 \le 1$,

$$\widetilde{\mathrm{RelDist}}(\alpha,\beta) = \frac{1}{\sqrt{\alpha/\beta}} - \sqrt{\alpha/\beta} \ge \frac{1}{\sqrt{\alpha/\beta_1}} - \sqrt{\alpha/\beta_1} = \widetilde{\mathrm{RelDist}}(\alpha,\beta_1),$$

as was to be shown. ∎

Remark: In Property 6 of Proposition 2.8, the assumption $\alpha\beta \ge 0$ is essential, since the inequality (2.17) is clearly violated if $\alpha < 0 < \alpha_1 < \beta_1 \le \beta$ and α_1 is sufficiently close to 0.

As before, let us relate Barlow-Demmel-Veselić relative distance to the common sense of relative perturbations.

Proposition 2.9 *Let* $\alpha, \beta \in \mathbb{R}$*. If* $0 \le \epsilon < 1$*, then*

$$\left|\frac{\beta}{\alpha} - 1\right| \le \epsilon \Rightarrow \widetilde{\mathrm{RelDist}}(\alpha,\beta) \le \frac{\epsilon}{\sqrt{1-\epsilon}}; \tag{2.19}$$

if $0 \le \epsilon < 2$*, then*

$$\widetilde{\mathrm{RelDist}}(\alpha,\beta) \le \epsilon \Rightarrow \max\left\{\left|\frac{\beta}{\alpha} - 1\right|, \left|\frac{\alpha}{\beta} - 1\right|\right\} \le \left(\frac{\epsilon}{2} + \sqrt{1 + \frac{\epsilon^2}{4}}\right)\epsilon. \tag{2.20}$$

Asymptotically,

$$\lim_{\beta\to\alpha} \frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{\left|\frac{\beta}{\alpha} - 1\right|} = 1,$$

thus (2.19) and (2.20) are at least asymptotically sharp.

Proof: The left-hand side of (2.19) implies $\beta = \alpha(1+\delta)$ for some $\delta \in \mathbb{R}$ with $|\delta| \le \epsilon$. So

$$\widetilde{\mathrm{RelDist}}(\alpha,\beta) = \frac{|\delta\alpha|}{\sqrt{\alpha^2(1+\delta)}} \le \frac{\epsilon}{\sqrt{1-\epsilon}},$$

as required. To prove (2.20), we set either $\xi = \alpha/\beta$ or $\xi = \beta/\alpha$. Since $\epsilon < 2$, $\xi > 0$. $\widetilde{\mathrm{RelDist}}(\alpha,\beta) \overset{\mathrm{def}}{=} \eta \le \epsilon$ gives

$$\frac{|\xi - 1|}{\sqrt{\xi}} = \eta \Rightarrow \xi^2 - (2+\eta^2)\xi + 1 = 0,$$

solving which yields

$$\xi = \frac{2+\eta^2 \pm \sqrt{(2+\eta^2)^2 - 4}}{2} = 1 + \left(\frac{\eta}{2} \pm \sqrt{1+\frac{\eta^2}{4}}\right)\eta.$$

Hence

$$|\xi - 1| \leq \left(\frac{\eta}{2} + \sqrt{1+\frac{\eta^2}{4}}\right)\eta \leq \left(\frac{\epsilon}{2} + \sqrt{1+\frac{\epsilon^2}{4}}\right)\epsilon$$

as was to be shown. ∎

Proposition 2.10 *Let $\widetilde{\beta} = \beta(1+\delta)$. Assume that $|\beta| \leq |\alpha|$ and $|\delta| \leq \epsilon < 1$, then*

$$\frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{1-\epsilon} + \frac{\epsilon}{1-\epsilon} \geq \widetilde{\mathrm{RelDist}}(\alpha,\widetilde{\beta}) \geq \frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{1+\epsilon} - \frac{\epsilon}{1+\epsilon}. \qquad (2.21)$$

Proof: Since $|\beta|(1-\epsilon) \leq |\widetilde{\beta}| \leq |\beta|(1+\epsilon)$ and $|\beta/\alpha| \leq 1$,

$$\begin{aligned}
\widetilde{\mathrm{RelDist}}(\alpha,\widetilde{\beta}) &= \frac{|\alpha-\widetilde{\beta}|}{\sqrt{|\alpha\widetilde{\beta}|}} \geq \frac{|\alpha-\beta|-|\delta\beta|}{\sqrt{|\alpha\widetilde{\beta}|}} \\
&\geq \frac{|\alpha-\beta|-\epsilon|\beta|}{\sqrt{|\alpha\beta|}\,(1+\epsilon)} \\
&= \frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{1+\epsilon} - \frac{\epsilon}{1+\epsilon}, \\
\widetilde{\mathrm{RelDist}}(\alpha,\widetilde{\beta}) &\leq \frac{|\alpha-\beta|+|\delta\beta|}{\sqrt{|\alpha\widetilde{\beta}|}} \\
&\leq \frac{|\alpha-\beta|+\epsilon|\beta|}{\sqrt{|\alpha\beta|}\,(1-\epsilon)} \\
&= \frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{1-\epsilon} + \frac{\epsilon}{1-\epsilon}.
\end{aligned}$$

as required. ∎

Proposition 2.10, in contrast to Proposition 2.3, only provides bounds on how $\widetilde{\mathrm{RelDist}}$ varies when one of its arguments smallest in magnitude is perturbed a little. Generally, we do not have a nice inequality like (2.11) for $\widetilde{\mathrm{RelDist}}$. Following the lines of the proof above, one can establish

$$\frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{1-\epsilon} + \frac{\epsilon}{1-\epsilon}\frac{|\alpha|+|\beta|}{\sqrt{|\alpha\beta|}} \geq \widetilde{\mathrm{RelDist}}(\alpha,\widetilde{\beta}) \geq \frac{\widetilde{\mathrm{RelDist}}(\alpha,\beta)}{1+\epsilon} - \frac{\epsilon}{1+\epsilon}\frac{|\alpha|+|\beta|}{\sqrt{|\alpha\beta|}},$$

where $\widetilde{\alpha} = \alpha(1+\delta_1)$ with $|\delta_1| \le \epsilon$. So the ratio $\frac{|\alpha|+|\beta|}{\sqrt{|\alpha\beta|}}$ which could be very large plays a crucial role.

Proposition 2.11 *For $\alpha, \beta \ge 0$,*

$$\widetilde{\mathrm{RelDist}}(\alpha^2, \beta^2) \ge 2\, \widetilde{\mathrm{RelDist}}(\alpha, \beta),$$

and the equality sign holds if and only if $\alpha = \beta$.

Proof: If either α or β is zero, no proof is required. Assume both are positive.

$$\widetilde{\mathrm{RelDist}}(\alpha^2, \beta^2) = \frac{\alpha+\beta}{\sqrt{\alpha\beta}} \frac{|\alpha-\beta|}{\sqrt{\alpha\beta}} \ge 2 \frac{|\alpha-\beta|}{\sqrt{\alpha\beta}} = 2\, \widetilde{\mathrm{RelDist}}(\alpha, \beta)$$

as was to be shown. ∎

Again there is no universal constant $c > 0$ so that $\widetilde{\mathrm{RelDist}}(\alpha, \beta)$ is bounded by $c \times \widetilde{\mathrm{RelDist}}(\alpha^2, \beta^2)$, unlike (2.13) in Proposition 2.4. One can always bound $\mathrm{RelDist}_p$ by $\widetilde{\mathrm{RelDist}}$, but not the other way around.

Proposition 2.12 *For $\alpha, \beta \in \mathbb{C}$,*

$$\mathrm{RelDist}_p(\alpha, \beta) \le 2^{-1/p}\, \widetilde{\mathrm{RelDist}}(\alpha, \beta),$$

and the equality sign holds if and only if $|\alpha| = |\beta|$.

Proof: Since

$$|\alpha|^p + |\beta|^p \ge 2\sqrt{|\alpha|^p |\beta|^p} = 2\left(\sqrt{|\alpha\beta|}\right)^p \Rightarrow \sqrt[p]{|\alpha|^p + |\beta|^p} \ge 2^{1/p}\sqrt{|\alpha\beta|},$$

from which the inequality follows. ∎

Proposition 2.12 is useful in that, as we will see later, any bound with $\widetilde{\mathrm{RelDist}}$ yields a bound with $\mathrm{RelDist}_p$. Now consider the same pairing problem for this newly-defined $\widetilde{\mathrm{RelDist}}$. First of all, the conclusion of Proposition 2.7 clearly remains valid if $\mathrm{RelDist}_2$ is replaced by $\widetilde{\mathrm{RelDist}}$ because of Property 5 in Proposition 2.8; second, with the help of Property 6 in Proposition 2.8 we can prove the same conclusion for $\widetilde{\mathrm{RelDist}}$ as that for $\mathrm{RelDist}_p$ in Proposition 2.5.

Proposition 2.13 *Under the conditions of Proposition 2.5, we have*

$$\max_{1\le i\le n} \widetilde{\mathrm{RelDist}}(\alpha_i, \widetilde{\alpha}_i) = \min_{\tau} \max_{1\le i\le n} \widetilde{\mathrm{RelDist}}(\alpha_i, \widetilde{\alpha}_{\tau(i)}),$$

where the minimization is taken over all permutations τ of $\{1, 2, \cdots, n\}$.

Remark. Proposition 2.13 may fail if not all α_i's and $\widetilde{\alpha}_j$'s are of the same sign. A *counterexample* is as follows: $n = 2$ and

$$\alpha_1 = -1 < \alpha_2 = 1 \quad \text{and} \quad \widetilde{\alpha}_1 = \frac{1}{4} < \widetilde{\alpha}_2 = 2.$$

We have showed that (2.15) cannot holds generally. In what follows, we will see that $\widetilde{\text{RelDist}}$ can do better.

Lemma 2.1 *Let $0 < \alpha_1 \le \alpha_2$ and $0 < \widetilde{\alpha}_1 \le \widetilde{\alpha}_2$. Then*

$$\left[\widetilde{\text{RelDist}}(\alpha_1, \widetilde{\alpha}_1)\right]^2 + \left[\widetilde{\text{RelDist}}(\alpha_2, \widetilde{\alpha}_2)\right]^2 \le \left[\widetilde{\text{RelDist}}(\alpha_1, \widetilde{\alpha}_2)\right]^2 + \left[\widetilde{\text{RelDist}}(\alpha_2, \widetilde{\alpha}_1)\right]^2,$$

or in another word,

$$\frac{(\widetilde{\alpha}_1 - \alpha_1)^2}{\widetilde{\alpha}_1\alpha_1} + \frac{(\widetilde{\alpha}_2 - \alpha_2)^2}{\widetilde{\alpha}_2\alpha_2} \le \frac{(\widetilde{\alpha}_2 - \alpha_1)^2}{\widetilde{\alpha}_2\alpha_1} + \frac{(\widetilde{\alpha}_1 - \alpha_2)^2}{\widetilde{\alpha}_1\alpha_2},$$

and the equality sign holds if and only if either $\alpha_1 = \alpha_2$ or $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

Proof: Complicated algebraic manipulations show that

$$\begin{aligned}\widetilde{\alpha}_1\alpha_1\widetilde{\alpha}_2\alpha_2 \left(\frac{(\widetilde{\alpha}_1 - \alpha_1)^2}{\widetilde{\alpha}_1\alpha_1} + \frac{(\widetilde{\alpha}_2 - \alpha_2)^2}{\widetilde{\alpha}_2\alpha_2} - \frac{(\widetilde{\alpha}_2 - \alpha_1)^2}{\widetilde{\alpha}_2\alpha_1} - \frac{(\widetilde{\alpha}_1 - \alpha_2)^2}{\widetilde{\alpha}_1\alpha_2}\right) \\ = \quad -(\alpha_2 - \alpha_1)(\widetilde{\alpha}_2 - \widetilde{\alpha}_1)(\widetilde{\alpha}_1\widetilde{\alpha}_2 + \alpha_1\alpha_2) \le 0,\end{aligned}$$

and the equality sign holds if and only if either $\alpha_1 = \alpha_2$ or $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$. ∎

Armed with Lemma 2.1, by following the proof of Proposition 2.5, one can show that

Proposition 2.14 *Let $\{\alpha_1, \cdots, \alpha_n\}$ and $\{\widetilde{\alpha}_1, \cdots, \widetilde{\alpha}_n\}$ be two sequences of n positive numbers ordered ascendingly (descendingly) as in (2.14). Then*

$$\sum_{i=1}^{n} \left[\widetilde{\text{RelDist}}(\alpha_i, \widetilde{\alpha}_i)\right]^2 = \min_{\tau} \sum_{i=1}^{n} \left[\widetilde{\text{RelDist}}(\alpha_i, \widetilde{\alpha}_{\tau(i)})\right]^2,$$

where the minimization is taken over all permutations τ of $\{1, 2, \cdots, n\}$.

Remark. It is clear to see that the conclusion of Proposition 2.14 remain valid if we weaken the conditions by only assuming that α_i's and $\widetilde{\alpha}_j$'s are nonnegative *and* the number of zeros in α_i's equals that in $\widetilde{\alpha}_j$'s. Proposition 2.14 may fail if not all α_i's and $\widetilde{\alpha}_j$'s are of the same sign. Here is a *counterexample*: $n = 2$ and

$$\alpha_1 = -2 < \alpha_2 = 1 \quad \text{and} \quad \widetilde{\alpha}_1 = 1 < \widetilde{\alpha}_2 = 2.$$

2.3 Are $\mathrm{RelDist}_p$ and $\widetilde{\mathrm{RelDist}}$ Metrics?

Let $\mathbb{X}$ be a space. Recall that a function $d : \mathbb{X} \times \mathbb{X} \mapsto [0, \infty)$ is called a *metric* if it has the following three properties: for $\alpha, \beta, \gamma \in \mathbb{X}$

1. $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;
2. $d(\alpha, \beta) = d(\beta, \alpha)$;
3. $d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)$.

This definition excludes immediately the possibility that $\widetilde{\mathrm{RelDist}}$ is a metric on $\mathbb{C}$, nor even on $\mathbb{R}$ since $\widetilde{\mathrm{RelDist}}(\alpha, 0) = \infty$ for $\alpha \neq 0$. To get around this, we, as any mathematician would do, extend this definition of a metric by calling $d : \mathbb{X} \times \mathbb{X} \mapsto [0, \infty]$ a *generalized metric* if it possesses the above three properties.

Now take a look at Propositions 2.1 and 2.8. We see that the functions $\mathrm{RelDist}_p$ and $\widetilde{\mathrm{RelDist}}$ on $\mathbb{C} \times \mathbb{C}$ satisfy the first two of the definition of a (generalized) metric. Naturally, we would like to ask: Is $\mathrm{RelDist}_p$ a metric on $\mathbb{C}$? and is $\widetilde{\mathrm{RelDist}}$ a generalized metric on $\mathbb{C}$? Or, equivalently, we may ask if for $\alpha, \beta, \gamma \in \mathbb{C}$

$$\mathrm{RelDist}_p(\alpha, \gamma) \leq \mathrm{RelDist}_p(\alpha, \beta) + \mathrm{RelDist}_p(\beta, \gamma)? \tag{2.22}$$

$$\widetilde{\mathrm{RelDist}}(\alpha, \gamma) \leq \widetilde{\mathrm{RelDist}}(\alpha, \beta) + \widetilde{\mathrm{RelDist}}(\beta, \gamma)? \tag{2.23}$$

At this point, we are able to formulate our incomplete answers into Proposition 2.15. Since the proof is quite long and tedious, we leave it to Appendixs A and B. Denote

$$\mathbb{R}_{\geq 0} \stackrel{\mathrm{def}}{=} [0, \infty) \quad \text{and} \quad \mathbb{R}_+ \stackrel{\mathrm{def}}{=} (0, \infty).$$

Proposition 2.15

1. *(2.22) holds for all $\alpha, \beta, \gamma \geq 0$ and $1 \leq p \leq \infty$, and thus $\mathrm{RelDist}_p$ is a metric on $\mathbb{R}_{\geq 0}$;*
2. *(2.22) with $p = 1, 2$ or ∞ holds for $\alpha, \beta, \gamma \in \mathbb{R}$, and thus $\mathrm{RelDist}_1$, $\mathrm{RelDist}_2$ and $\mathrm{RelDist}_\infty$ are metrics on $\mathbb{R}$;*
3. *(2.23) holds for $\alpha, \beta, \gamma \geq 0$, but not on whole $\mathbb{R}$, and thus $\widetilde{\mathrm{RelDist}}$ is a generalized metric on $\mathbb{R}_{\geq 0}$, but not on $\mathbb{R}$ nor $\mathbb{C}$.*

Still the question whether $\mathrm{RelDist}_p$ is a metric on $\mathbb{C}$ is open.

3 Summary of Results

To help the reader to grasp quickly what we have accomplished in this paper, we give here a table to summarize partially the simplified versions of our theorems in comparison with their corresponding well-known theorems in literature. Full statement of these theorems and their stronger versions will be done in §5 and §6. More results will be discussed in §7. Before we present the table, let us stick to some notation: A, $\widetilde{A} \in \mathbb{C}^{n\times n}$, and

$$\lambda(A) = \{\lambda_1, \cdots, \lambda_n\} \quad \text{and} \quad \lambda(\widetilde{A}) = \{\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_n\}; \tag{3.1}$$

B, $\widetilde{B} \in \mathbb{C}^{m\times n}$, and

$$\sigma(B) = \{\sigma_1, \cdots, \sigma_n\} \quad \text{and} \quad \sigma(\widetilde{B}) = \{\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_n\}. \tag{3.2}$$

In the table, τ always stands for some permutation of $\{1, 2, \cdots, n\}$; σ_i's and $\widetilde{\sigma}_j$'s are assumed in descending order, i.e.,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0, \quad \widetilde{\sigma}_1 \geq \widetilde{\sigma}_2 \geq \cdots \geq \widetilde{\sigma}_n \geq 0; \tag{3.3}$$

Whenever, all λ_i's and $\widetilde{\lambda}_j$'s are real, we also require

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \quad \widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \cdots \geq \widetilde{\lambda}_n. \tag{3.4}$$

In Table 3.1, each row consists of four boxes. The first one describes conditions under which the inequality in the second box holds; the third one states, besides these in the first one, additional conditions in order for the inequality in the fourth box to be true.

Table 3.1. Summary: (i) Hoffman and Wielandt Type Theorems

Classical Bounds		New Relative Bounds	
A and $\widetilde{A}$ normal	$\sqrt{\sum_{i=1}^{n} \lvert\lambda_i - \widetilde{\lambda}_{\tau(i)}\rvert^2} \le \|\widetilde{A} - A\|_F$ (Theorem 4.1)	$\widetilde{A} = D_1^* A D_2$	$\sqrt{\sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} \le \min\Big\{ \sqrt{\|I - D_1\|_F^2 + \|I - D_2^{-1}\|_F^2}, \sqrt{\|I - D_1^{-1}\|_F^2 + \|I - D_2\|_F^2}\Big\}$ (Theorem 6.2)
A and $\widetilde{A}$ Hermitian	$\sqrt{\sum_{i=1}^{n} \lvert\lambda_i - \widetilde{\lambda}_i\rvert^2} \le \|\widetilde{A} - A\|_F$ (Theorems 4.1 and 4.3)	$\widetilde{A} = D^* A D$	$\sqrt{\sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} \le \sqrt{\|I - D\|_F^2 + \|I - D^{-1}\|_F^2}$ (Theorem 6.3)
A and $\widetilde{A}$ Definite	$\sqrt{\sum_{i=1}^{n} \lvert\lambda_i - \widetilde{\lambda}_i\rvert^2} \le \|\widetilde{A} - A\|_F$ (Theorems 4.1 and 4.3)	$\widetilde{A} = D^* A D$	$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\mathrm{RelDist}}(\lambda_i, \widetilde{\lambda}_i)\right]^2} \le \|D^* - D^{-1}\|_F$ (Theorem 5.1)
$A = X\Lambda X^{-1}$, $\widetilde{A} = \widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}$	$\sqrt{\sum_{i=1}^{n} \lvert\lambda_i - \widetilde{\lambda}_{\tau(i)}\rvert^2} \le \kappa(X)\kappa(\widetilde{X})\|\widetilde{A} - A\|_F$ (Theorem 4.2)	$\widetilde{A} = D_1^* A D_2$	$\sqrt{\sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} \le \kappa(X)\kappa(\widetilde{X}) \min\{ \sqrt{\|I - D_1\|_F^2 + \|I - D_2^{-1}\|_F^2}, \sqrt{\|I - D_1^{-1}\|_F^2 + \|I - D_2\|_F^2}\}$ (Theorem 6.1)
B and $\widetilde{B}$	$\sqrt{\sum_{i=1}^{n} \lvert\sigma_i - \widetilde{\sigma}_i\rvert^2} \le \|\widetilde{B} - B\|_F$ (Theorem 4.7)	$\widetilde{B} = D_1^* B D_2$	$\sqrt{\sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\sigma_i, \widetilde{\sigma}_{\tau(i)})\right]^2} \le \frac{1}{\sqrt{2}} \left[\|I - D_1\|_F^2 + \|I - D_1^{-1}\|_F^2 + \|I - D_2\|_F^2 + \|I - D_2^{-1}\|_F^2\right]^{1/2}$ (Theorem 6.7)
B and $\widetilde{B}$	$\sqrt{\sum_{i=1}^{n} \lvert\sigma_i - \widetilde{\sigma}_i\rvert^2} \le \|\widetilde{B} - B\|_F$ (Theorem 4.7)	$\widetilde{B} = D_1^* B D_2$	$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} \le \frac{\|D_1^* - D_1^{-1}\|_F + \|D_2^* - D_2^{-1}\|_F}{2}$ (Theorem 5.2)

Table 3.1. Summary (continued): (ii) Weyl-Lidskii Type Theorems

Classical Bounds		New Relative Bounds	
A and $\widetilde{A}$ Hermitian	$\lvert\lambda_i - \widetilde{\lambda}_i\rvert \le \Vert\widetilde{A} - A\Vert_2$ (Theorem 4.3)	$\widetilde{A} = D^*AD$	$\text{RelDist}_\infty(\lambda_i, \widetilde{\lambda}_i) \le \Vert I - D^*D\Vert_2$, $\widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \le \frac{\Vert I - D^*D\Vert_2}{\sigma_{\min}(D)}$ (Cf. (7.3) and (7.4))
A and $\widetilde{A}$ Definite	$\lvert\lambda_i - \widetilde{\lambda}_i\rvert \le \Vert\widetilde{A} - A\Vert_2$ (Theorem 4.3)	$\widetilde{A} = D^*AD$	$\widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \le \Vert D^* - D^{-1}\Vert_2$ (Theorem5.1)
$A = X\Lambda X^{-1}$, $\widetilde{A} = \widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}$ Λ and $\widetilde{\Lambda}$ real nonnegative	$\lvert\lambda_i - \widetilde{\lambda}_i\rvert$ $\le \kappa(X)\kappa(\widetilde{X})\Vert\widetilde{A} - A\Vert_2$ (Theorems 4.4 and 4.5)	$\widetilde{A} = D_1^*AD_2$	$\text{RelDist}_p(\lambda_i, \widetilde{\lambda}_i)$ $\le \kappa(X)\kappa(\widetilde{X}) \min \{$ $\sqrt[q]{\Vert I - D_1^*\Vert_2^q + \Vert I - D_2^{-1}\Vert_2^q}$, $\sqrt[q]{\Vert I - D_1^{-*}\Vert_2^q + \Vert I - D_2\Vert_2^q}\}$ (Theorem 6.4)
B and $\widetilde{B}$	$\lvert\sigma_i - \widetilde{\sigma}_i\rvert \le \Vert\widetilde{B} - B\Vert_2$ (Theorem 4.7)	$\widetilde{B} = D_1^*BD_2$	$\text{RelDist}_p(\sigma_i, \widetilde{\sigma}_i) \le \min \{$ $\sqrt[q]{\Vert I - D_1^{-1}\Vert_2^q + \Vert I - D_2\Vert_2^q}$, $\sqrt[q]{\Vert I - D_1\Vert_2^q + \Vert I - D_2^{-1}\Vert_2^q}\}$ (Theorem 6.8)
B and $\widetilde{B}$	$\lvert\sigma_i - \widetilde{\sigma}_i\rvert \le \Vert\widetilde{B} - B\Vert_2$ (Theorem 4.7)	$\widetilde{B} = D_1^*BD_2$	$\widetilde{\text{RelDist}}(\sigma_i, \widetilde{\sigma}_i)$ $\le \frac{\Vert D_1^* - D_1^{-1}\Vert_2 + \Vert D_2^* - D_2^{-1}\Vert_2}{2}$ (Theorem 5.2)

Table 3.1. Summary (continued): (iii) A Bauer-Fike Type Theorem

Classical Bounds		New Relative Bounds	
$A = X\Lambda X^{-1}$	$\forall\widetilde{\lambda} \in \lambda(\widetilde{A})$, $\exists\lambda \in \lambda(A)$, such that $\lvert\widetilde{\lambda} - \lambda\rvert \le \kappa(X)\Vert\widetilde{A} - A\Vert_2$ (Theorem 4.6)	Either $\widetilde{A} = AD$ or $\widetilde{A} = DA$.	$\forall\widetilde{\lambda} \in \lambda(\widetilde{A})$, $\exists\lambda \in \lambda(A)$, such that $\frac{\lvert\widetilde{\lambda} - \lambda\rvert}{\lvert\lambda\rvert} \le \kappa(X)\Vert I - D\Vert_2$ (Theorem 6.6)

Finally, let's consider the graded case for which we will use $H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$ for two $n \times n$ graded nonnegative definite Hermitian matrices with A nonsingular and $\|A^{-1}\|_2\|\Delta A\|_2 < 1$, where $\Delta A \stackrel{\rm def}{=} \widetilde{A} - A$, and $G = BD$ and $\widetilde{G} = \widetilde{B}D$ for two $m \geq n$ graded matrices whose singular values are of interest. Also it is required that B is nonsingular and $\|B^{-1}\|_2\|\Delta B\|_2 < 1$ where $\Delta B \stackrel{\rm def}{=} \widetilde{B} - B$. Denote

$$\lambda(H) = \{\lambda_1, \cdots, \lambda_n\} \quad \text{and} \quad \lambda(\widetilde{H}) = \{\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_n\};$$

and

$$\sigma(G) = \{\sigma_1, \cdots, \sigma_n\} \quad \text{and} \quad \sigma(\widetilde{G}) = \{\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_n\},$$

and arrange them in the order prescribed by (3.3) and (3.4). Set

$$E_A \stackrel{\rm def}{=} A^{-1/2}(\Delta A)A^{-1/2} \quad \text{and} \quad E_B \stackrel{\rm def}{=} (\Delta B)B^{-1}.$$

Table 3.1. Summary (continued): (iv) Theorems for Graded Matrices

Classical Bounds		New Relative Bounds	
H and $\widetilde{H}$ Definite	$\sqrt{\sum_{i=1}^{n} \lvert\lambda_i - \widetilde{\lambda}_i\rvert^2} \leq \|\widetilde{H} - H\|_F$ (Theorem 4.1 and 4.3)	$H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$	$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i)\right]^2} \leq \|(I+E_A)^{1/2} - (I+E_A)^{-1/2}\|_F$ (Theorem 5.4)
H and $\widetilde{H}$ Definite	$\lvert\lambda_i - \widetilde{\lambda}_i\rvert \leq \|\widetilde{H} - H\|_2$ (Theorem 4.3)	$H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$	$\widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \leq \|(I+E_A)^{1/2} - (I+E_A)^{-1/2}\|_2$ (Theorem 5.4)
G and $\widetilde{G}$	$\sqrt{\sum_{i=1}^{n} \lvert\sigma_i - \widetilde{\sigma}_i\rvert^2} \leq \|\widetilde{G} - G\|_F$ (Theorem 4.7)	$G = BD$ and $\widetilde{G} = \widetilde{B}D$	$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\text{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} \leq \frac{\|(I+E_B)^* - (I+E_B)^{-1}\|_F}{2}$ (Theorem 5.3)
G and $\widetilde{G}$	$\lvert\sigma_i - \widetilde{\sigma}_i\rvert \leq \|\widetilde{G} - G\|_2$ (Theorem 4.7)	$G = BD$ and $\widetilde{G} = \widetilde{B}D$	$\widetilde{\text{RelDist}}(\sigma_i, \widetilde{\sigma}_i) \leq \frac{\|(I+E_B)^* - (I+E_B)^{-1}\|_2}{2}$ (Theorem 5.3)

4 Known Perturbation Theorems for Eigenvalue and Singular Value Variations

In this section, we will briefly review a few most celebrated theorems for eigenvalue and singular value variations which will be generalized. Most of this theorems can be found in Bhatia [3], Golub and Van Loan [14], Parlett [28] and Stewart and Sun [30]. Notation introduced in §3 will be followed strictly. Hoffman and Wielandt [16] proved

Theorem 4.1 (Hoffman-Wielandt) *If A and $\widetilde{A}$ are normal, then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\sqrt{\sum_{i=1}^{n} |\lambda_i - \widetilde{\lambda}_{\tau(i)}|^2} \le \|\widetilde{A} - A\|_F.$$

For a nonsingular matrix $X \in \mathbb{C}^{n\times n}$, the (spectral) condition number $\kappa(X)$ is defined as

$$\kappa(X) \stackrel{\mathrm{def}}{=} \|X\|_2 \|X^{-1}\|_2.$$

Theorem 4.1 was generalized by Sun [33] and Zhang [37] to two diagonalizable matrices.

Theorem 4.2 (Sun-Zhang) *Assume that both A and $\widetilde{A}$ are diagonalizable and admit the following decompositions*

$$A = X\Lambda X^{-1} \quad \textit{and} \quad \widetilde{A} = \widetilde{X}\widetilde{\Lambda}\widetilde{X}^{-1}, \tag{4.1}$$

where X and $\widetilde{X}$ are nonsingular and

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \quad \textit{and} \quad \widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_n). \tag{4.2}$$

Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that

$$\sqrt{\sum_{i=1}^{n} |\lambda_i - \widetilde{\lambda}_{\tau(i)}|^2} \le \kappa(X)\kappa(\widetilde{X})\|\widetilde{A} - A\|_F.$$

We will consider *unitarily invariant norms* $|||\cdot|||$ of matrices. In this we follow Mirsky [27] and Stewart & Sun [30]. To say that the norm is *unitarily invariant* on $\mathbb{C}^{m\times n}$ means that it satisfies, besides the usual properties of any norm, also

1. $|||UXV||| = |||X|||$, for any $U \in \mathbb{U}_m$, and $V \in \mathbb{U}_n$;
2. $|||X||| = \|X\|_2$, for any $X \in \mathbb{C}^{m\times n}$ with $\mathrm{rank} X = 1$.

Two unitarily invariant norms used frequently are the *spectral norm* $\|\cdot\|_2$ and the *Frobenius norm* $\|\cdot\|_F$. Let $|||\cdot|||$ be a unitarily invariant norm living in some matrix space. the following inequalities [30, p. 80] will be employed very frequently in the rest this paper.

$$|||XY||| \leq \|X\|_2\, |||Y||| \quad \text{and} \quad |||YZ||| \leq |||Y|||\, \|Z\|_2.$$

Theorem 4.3 *Suppose that A and $\widetilde{A}$ are both Hermitian, and that (3.4) holds. Then for any unitarily invariant norm $|||\cdot|||$*

$$\left|\left|\left|\operatorname{diag}(\lambda_1 - \widetilde{\lambda}_1, \cdots, \lambda_n - \widetilde{\lambda}_n)\right|\right|\right| \leq \left|\left|\left|A - \widetilde{A}\right|\right|\right|. \tag{4.3}$$

The inequality (4.3) was proved by Weyl [35] for the spectral norm, by Loewner [24] and as a corollary of Hoffman-Wielandt theorem [16] for the Frobenius norm and by Lidskii [23], Wielandt [36] and Mirsky [27] for all unitarily invariant norms. Neither Lidskii nor Wielandt mentioned explicitly (4.3) which was done by Mirsky [27]. For more detail, the reader is referred to Bhatia [3]. Theorem 4.3 has been generalized in many aspects. The following theorem is due to Bhatia, Davis and Kittaneh [4].

Theorem 4.4 (Kahan, Bhatia, Davis and Kittaneh) *To the hypotheses of Theorem 4.2 adds this: all λ_i's and $\widetilde{\lambda}_j$'s are real and are arranged descendingly as in (3.4). Then for any unitarily invariant norm $|||\cdot|||$*

$$\left|\left|\left|\operatorname{diag}(\lambda_1 - \widetilde{\lambda}_1, \cdots, \lambda_n - \widetilde{\lambda}_n)\right|\right|\right| \leq \kappa(X)\kappa(\widetilde{X}) \left|\left|\left|A - \widetilde{A}\right|\right|\right|. \tag{4.4}$$

The inequality (4.4) was proved by Kahan [19] for the spectral norm, as a corollary of Sun-Zhang theorem [33, 37] for the Frobenius norm. In another aspect, the inequality (4.3) for the spectral norm was generalized to ℓ_p operator norm. The *p-Hölder norm* of a vector $x = (\xi_i) \in \mathbb{C}^n$ is defined by

$$\|x\|_p \stackrel{\text{def}}{=} \sqrt[p]{\sum_{i=1}^{n} |\xi_i|^p}.$$

The ℓ_p-operator norm of a matrix $X \in \mathbb{C}^{n\times n}$ is defined by

$$\|X\|_p \stackrel{\text{def}}{=} \max_{\|x\|_p=1} \|Xx\|_p.$$

If X is nonsingular, its ℓ_p *condition number* is defined by

$$\kappa_p(X) \stackrel{\text{def}}{=} \|X\|_p \|X^{-1}\|_p.$$

Clearly, $\kappa_2(X) = \kappa(X)$, the (spectral) condition number. The following theorem is due to Li [21, pp. 225–226].

Theorem 4.5 (Li) *Under the conditions of Theorem 4.4. Then*

$$\max_{1\le i\le n} |\lambda_i - \widetilde{\lambda}_i| \le \kappa_p(X)\kappa_p(\widetilde{X})\|A-\widetilde{A}\|_p,$$

where $1 \le p \le \infty$.

Generally, if one of A and $\widetilde{B}$ is diagonalizable, we have the following result due to Bauer and Fike[1] [2].

Theorem 4.6 (Bauer-Fike) *Assume* A *is diagonalizable, i.e.,*

$$A = X\Lambda X^{-1}, \quad \textit{where } \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

Then for any $\widetilde{\lambda} \in \lambda(\widetilde{A})$, *there exists a* $\lambda \in \lambda(A)$ *such that*

$$|\widetilde{\lambda} - \lambda| \le \kappa(X)\|\widetilde{A} - A\|_2. \tag{4.5}$$

Regarding singular value perturbations, the following theorem was established in Mirsky [27], based on Lidskii [23] and Wielandt [36].

Theorem 4.7 *Arrange the singular values of* B *and* $\widetilde{B}$ *in descending order as in (3.3). Then for any unitarily invariant norm* $|||\cdot|||$

$$|||\mathrm{diag}(\sigma_1 - \widetilde{\sigma}_1, \cdots, \sigma_n - \widetilde{\sigma}_n)||| \le \left|\left|\left| B - \widetilde{B} \right|\right|\right|. \tag{4.6}$$

[1] One can prove a slightly more stronger inequality than (4.5)

$$|\widetilde{\lambda} - \lambda| \le \|X^{-1}(\widetilde{A} - A)X\|_2.$$

5 Statement of Theorems with $\widetilde{\text{RelDist}}$: Nonnegative Definite Matrices

In this section, we devote our attention to the relative perturbation theory for eigenvalues of nonnegative definite matrices, including singular value problems. We will consider the following problems:

- **Eigenvalue problems:**
  1. A and $\widetilde{A} = D^* A D$ with A nonnegative definite and D being close to some unitary matrix;
  2. $H = D^* A D$ and $\widetilde{H} = D^* \widetilde{A} D$ with both A and $\widetilde{A}$ positive definite and $\|A^{-1}\|_2 \|\widetilde{A} - A\|_2 < 1$, where D is some square matrix.
- **Singular value problems:**
  1. B and $\widetilde{B} = D_1^* B D_2$ with D_1 and D_2 being close to some unitary matrices of suitable dimensions;
  2. $G = BD$ and $\widetilde{G} = \widetilde{B} D$ with both B and $\widetilde{B}$ nonsingular and $\|B^{-1}\|_2 \|\widetilde{B} - B\|_2 < 1$, where D is some square matrix.

Theorems presented here are often better than these in the next section when applying to nonnegative definite matrices. We will make this more concrete in the coming section.

5.1 Eigenvalue Variations for A and $\widetilde{A} = D^* A D$

Theorem 5.1 *Let A and $\widetilde{A} = D^* A D$ be two $n \times n$ Hermitian matrices, where D is nonsingular. Denote their eigenvalues as in (3.1) and arrange them descendingly as described in (3.4). Assume that A is nonnegative definite. Then*

$$\max_{1 \le i \le n} \widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \le \|D^* - D^{-1}\|_2, \tag{5.1}$$

$$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i)\right]^2} \le \|D^* - D^{-1}\|_F. \tag{5.2}$$

It is trivial to relate the right-hand sides of the inequalities (5.1) and (5.2) to the singular values of D. In fact, let SVD of D be

$$D = U_{\rm d} \Sigma_{\rm d} V_{\rm d}^*. \tag{5.3}$$

One has for any unitarily invariant norm $|||\cdot|||$

$$|||D^* - D^{-1}||| = |||V_{\rm d}(\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}) U_{\rm d}^*||| = |||\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}|||.$$

Another point we would like to make is that A and D^*AD have the same rank, or in another word, A and D^*AD have the same number of zero eigenvalues. In order for the inequalities (5.2) and (5.1) to be true, 0 eigenvalues, if any, must be always paired with 0 ones.

5.2 Singular Value Variations for B and $\widetilde{B} = D_1^* B D_2$

Theorem 5.2 *Let B and $\widetilde{B} = D_1^* B D_2$ be two $m \times n$ matrices, where D_1 and D_2 are square and nonsingular. Denote their singular values as in (3.2) and arrange them as in (3.3). Then*

$$\max_{1 \le i \le n} \widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i) \le \frac{1}{2}\left(\|D_1^* - D_1^{-1}\|_2 + \|D_2^* - D_2^{-1}\|_2\right), \tag{5.4}$$

$$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} \le \frac{1}{2}\left(\|D_1^* - D_1^{-1}\|_F + \|D_2^* - D_2^{-1}\|_F\right). \tag{5.5}$$

Now, Let's briefly mention a possible application of Theorem 5.2. It has something to do with *deflation* in computing the singular value systems of a bidiagonal matrix. For more details, the reader is referred to [6, 8, 10, 26]. We formulate the application into a corollary.

Corollary 5.1 *Assume in Theorem 5.2, one of the D_1 and D_2 is an identity matrix and the other takes the form*

$$D = \begin{pmatrix} I & X \\ & I \end{pmatrix},$$

where X is a matrix of suitable dimensions. With the notation of Theorem 5.2, we have

$$\max_{1 \le i \le n} \widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i) \le \frac{1}{2}\|X\|_2, \tag{5.6}$$

$$\sqrt{\sum_{i=1}^{n} \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} \le \frac{1}{\sqrt{2}}\|X\|_F. \tag{5.7}$$

Proof: Notice that

$$D^* - D^{-1} = \begin{pmatrix} I & \\ X^* & I \end{pmatrix} - \begin{pmatrix} I & -X \\ & I \end{pmatrix} = \begin{pmatrix} & X \\ X^* & \end{pmatrix},$$

and thus $\|D^* - D^{-1}\|_2 = \|X\|_2$ and $\|D^* - D^{-1}\|_F = \sqrt{2}\|X\|_F$. ∎

It was proved by Eisenstat and Ipsen [10] that

$$|\widetilde{\sigma}_i - \sigma_i| \le \|X\|_2 \sigma_i, \quad \text{or equivalently} \quad \left|\frac{\widetilde{\sigma}_i}{\sigma_i} - 1\right| \le \|X\|_2. \tag{5.8}$$

So as long as $\widetilde{\sigma}_i$ and σ_i are of the similar magnitude which is guaranteed if $\|X\|_2$ is small, our inequality (5.6) is sharper by a factor 1/2. As a matter of fact, it follows from (5.6) and Proposition 2.9 that if $\|X\|_2 \le 4$ then

$$\left|\frac{\widetilde{\sigma}_i}{\sigma_i} - 1\right| \le \left(\frac{\|X\|_2}{4} + \sqrt{1 + \frac{\|X\|_2^2}{16}}\right)\frac{\|X\|_2}{2} = \frac{\|X\|_2}{2} + O(\|X\|_2^2).$$

Our inequality (5.7) is the first of its kind.

5.3 Graded Matrices

Theorem 5.3 *Let $G = BD$ and $\widetilde{G} = \widetilde{B}D$ be two $n \times n$ matrices, where B and $\widetilde{B}$ are nonsingular, and let $\Delta B = \widetilde{B} - B$. Denote*

$$\sigma(G) = \{\sigma_1, \cdots, \sigma_n\} \quad \textit{and} \quad \sigma(\widetilde{G}) = \{\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_n\},$$

and arrange them descendingly as in (3.3). If $\|\Delta B\|_2 \|B^{-1}\|_2 < 1$, then

$$\begin{aligned}
&\max_{1\le i\le n} \widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i) \\
&\le \frac{1}{2}\left\|(I + (\Delta B)B^{-1})^* - (I + (\Delta B)B^{-1})^{-1}\right\|_2 \\
&\le \left(\frac{\|(\Delta B)B^{-1} + B^{-*}(\Delta B)^*\|_2}{\|(\Delta B)B^{-1}\|_2} + \frac{\|(\Delta B)B^{-1}\|_2}{1 - \|(\Delta B)B^{-1}\|_2}\right)\frac{\|(\Delta B)B^{-1}\|_2}{2} \\
&\le \left(1 + \frac{1}{1 - \|B^{-1}\|_2\|\Delta B\|_2}\right)\frac{\|B^{-1}\|_2\|\Delta B\|_2}{2}, \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
&\sqrt{\sum_{i=1}^{n}\left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} \\
&\le \frac{1}{2}\left\|(I + (\Delta B)B^{-1})^* - (I + (\Delta B)B^{-1})^{-1}\right\|_F \\
&\le \left(\frac{\|(\Delta B)B^{-1} + B^{-*}(\Delta B)^*\|_F}{\|(\Delta B)B^{-1}\|_F} + \frac{\|(\Delta B)B^{-1}\|_2}{1 - \|(\Delta B)B^{-1}\|_2}\right)\frac{\|(\Delta B)B^{-1}\|_F}{2} \\
&\le \left(1 + \frac{1}{1 - \|B^{-1}\|_2\|\Delta B\|_2}\right)\frac{\|B^{-1}\|_2\|\Delta B\|_F}{2}. \qquad (5.10)
\end{aligned}$$

Remark. It is interesting to notice that if $(\Delta B)B^{-1}$ is very skew, then $\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i) = o(\|(\Delta B)B^{-1}\|_2)$. Especially if $\|(\Delta B)B^{-1} + B^{-*}(\Delta B)^*\|_2 = O(\|(\Delta B)B^{-1}\|_2^2)$, then $\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i) = O(\|(\Delta B)B^{-1}\|_2^2)$ also.

Theorem 5.4 *Let $H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$ be two $n \times n$ nonnegative definite Hermitian matrices whose eigenvalues are*

$$\lambda(H) = \{\lambda_1, \cdots, \lambda_n\} \quad \textit{and} \quad \lambda(\widetilde{H}) = \{\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_n\}, \qquad (5.11)$$

and in descending order as in (3.4) and let $\Delta A = \widetilde{A} - A$. *If*

$$\|A^{-1}\|_2 \|\Delta A\|_2 < 1, \tag{5.12}$$

then

$$\begin{aligned}
&\max_{1 \le i \le n} \widetilde{\mathrm{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \\
&\quad \le \left\| (I + A^{-1/2}(\Delta A)A^{-1/2})^{1/2} - (I + A^{-1/2}(\Delta A)A^{-1/2})^{-1/2} \right\|_2 \\
&\quad \le \frac{\|A^{-1}\|_2 \|\Delta A\|_2}{\sqrt{1 - \|A^{-1}\|_2 \|\Delta A\|_2}},
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&\sqrt{\sum_{i=1}^{n} \left[\widetilde{\mathrm{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \right]^2} \\
&\quad \le \left\| (I + A^{-1/2}(\Delta A)A^{-1/2})^{1/2} - (I + A^{-1/2}(\Delta A)A^{-1/2})^{-1/2} \right\|_F \\
&\quad \le \frac{\|A^{-1}\|_2 \|\Delta A\|_F}{\sqrt{1 - \|A^{-1}\|_2 \|\Delta A\|_2}}.
\end{aligned} \tag{5.14}$$

6 Statement of Theorems with $\mathrm{RelDist}_p$

The rests of cases listed in §1.1, as well as singular value problems, will be treated here. To be specific, we will consider

- **Eigenvalue problems:**
  1. A and $\widetilde{A} = D^* A D$ for Hermitian case, where D is nonsingular and close to I or more generally to a unitary matrix;
  2. A and $\widetilde{A} = D_1^* A D_2$ for general diagonalizable case, where D_1 and D_2 are nonsingular and close to I or more generally to some unitary matrix;
- **Singular value problems:**
  1. B and $\widetilde{B} = D_1^* B D_2$, where D_1 and D_2 are nonsingular and close to I or more generally to two unitary matrices;

We retreat singular value problems for comparison purpose. As we will see soon that we will prove more nice inequalities for singular value variations, but these inequalities may be potentially less sharp than those in §5 for large perturbations. Brief comparisons among theorems in this section and these in the previous section will be given.

6.1 Eigenvalue Variations

The following theorem is a generalization of Theorems 4.1 and 4.2.

Theorem 6.1 *Assume that $n \times n$ matrix A is perturbed to $\widetilde{A} = D_1^* A D_2$ and both D_1 and D_2 are nonsingular. Assume also both A and $\widetilde{A}$ are diagonalizable and admit the decompositions as described in (4.1) and (4.2). Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$
\begin{aligned}
&\sqrt{\sum_{i=1}^{n} \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} \qquad (6.1)\\
\leq\;& \min\left\{ \|\widetilde{X}^{-1}\|_2 \|X\|_2 \sqrt{\|X^{-1}(I - D_2)\widetilde{X}\|_F^2 + \|X^{-1}(D_1^{-*} - I)\widetilde{X}\|_F^2}, \right.\\
&\qquad \left. \|X^{-1}\|_2 \|\widetilde{X}\|_2 \sqrt{\|\widetilde{X}^{-1}(I - D_1^*)X\|_F^2 + \|\widetilde{X}^{-1}(D_2^{-1} - I)X\|_F^2} \right\}\\
\leq\;& \kappa(X)\kappa(\widetilde{X}) \min\left\{ \sqrt{\|I - D_1\|_F^2 + \|I - D_2^{-1}\|_F^2}, \sqrt{\|I - D_1^{-1}\|_F^2 + \|I - D_2\|_F^2} \right\}.
\end{aligned}
$$

For any given $U \in \mathbb{U}_n$, $U\widetilde{A}U^* = (D_1U^*)^*AD_2U^*$ has the same eigenvalues as $\widetilde{A}$ does, and moreover from (4.1)

$$U\widetilde{A}U^* = (\widetilde{X}U^*)^{-1}\widetilde{\Lambda}\widetilde{X}U^*.$$

So applying Theorem 6.1 to matrices A and $U\widetilde{A}U^*$ leads to the following theorem which we will refer as Theorem 6.1s, where "s" is for indicating that it is stronger.

Theorem 6.1s *Let all conditions of Theorem 6.1 hold. Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\begin{aligned}
&\sqrt{\sum_{i=1}^{n}\left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} \qquad (6.2)\\
\leq\;& \kappa(X)\kappa(\widetilde{X}) \min_{U\in\mathbb{U}_n} \min\left\{\sqrt{\|U - D_1\|_F^2 + \|U^* - D_2^{-1}\|_F^2},\right.\\
&\left.\sqrt{\|U^* - D_1^{-1}\|_F^2 + \|U - D_2\|_F^2}\right\}.
\end{aligned}$$

Suppose now $A \in \mathbb{C}^n$ is an normal matrix, i.e., $A^*A = AA^*$. Perturb A to $\widetilde{A} = D_1^*AD_2$. The question is: *When is $\widetilde{A}$ also normal?* This is a rather interesting question, and an instant answer is that $\widetilde{A}$ *is normal provided*

$$D_2^*A^*D_1D_1^*AD_2 = D_1^*AD_2D_2^*A^*D_1.$$

However, this condition is, perhaps. too general to be useful. I do not know how to approach this problem yet and therefore this question will not be addressed further in what follows. On the other hand, if we happen to know that $\widetilde{A}$ is also normal, the following theorem, as a corollary of Theorem 6.1, indicates that the eigenvalues of A and $\widetilde{A}$ agrees to high relative accuracy.

Theorem 6.2 *Let A and $\widetilde{A} = D_1^*AD_2$ be two $n \times n$ normal matrixes, where D_1 and D_2 are nonsingular. Denote their eigenvalues as in (3.1). Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\begin{aligned}
&\sqrt{\sum_{i=1}^{n}\left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} \qquad (6.3)\\
\leq\;& \min_{U\in\mathbb{U}_n} \min\left\{\sqrt{\|U - D_1\|_F^2 + \|U^* - D_2^{-1}\|_F^2}, \sqrt{\|U^* - D_1^{-1}\|_F^2 + \|U - D_2\|_F^2}\right\}.
\end{aligned}$$

We happen to know how to solve the *minimization* problem: find a $U_0 \in \mathbb{U}_n$ such that for any unitarily invariant norm $|||\cdot|||$

$$\min_{U\in\mathbb{U}_n} |||U - D||| = |||U_0 - D||| \quad \text{and} \min_{U\in\mathbb{U}_n} \left|\left|\left|U^* - D^{-1}\right|\right|\right| = \left|\left|\left|U_0^* - D^{-1}\right|\right|\right|. \quad (6.4)$$

in terms of the singular value decomposition (SVD) of D. As a matter of fact, let SVD of D be given in (5.3). It follows from Theorem 4.7 that

$$|||U - D||| \geq |||I - \Sigma_{\rm d}||| \quad \text{and} \quad \left|\left|\left|U^* - D^{-1}\right|\right|\right| \geq \left|\left|\left|I - \Sigma_{\rm d}^{-1}\right|\right|\right|. \tag{6.5}$$

Fortunately, there is one $U_0 \stackrel{\rm def}{=} U_{\rm d} V_{\rm d}^*$ which realizes the two equality signs. Theorem 6.2, now applying to Hermitian matrices, leads to

Theorem 6.3 *Let A and $\widetilde{A} = D^* A D$ be two $n \times n$ Hermitian matrices, where D is nonsingular. Denote their eigenvalues as in (3.1). Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\begin{aligned}\sqrt{\sum_{i=1}^n \left[\text{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2} &\leq \min_{U \in \mathbb{U}_n} \sqrt{\|U - D\|_F^2 + \|U^* - D^{-1}\|_F^2} \\ &= \sqrt{\|I - \Sigma_{\rm d}\|_F^2 + \|I - \Sigma_{\rm d}^{-1}\|_F^2}.\end{aligned} \tag{6.6}$$

It is worth mentioning that the permutation τ in Theorem 6.3 may not be the identity one, assuming eigenvalues are ordered in the way of (3.4). However, one can always choose a τ such that zeros are matched to zeros, negative eigenvalues to negative ones and positive ones to positive ones (Cf. Propositions 2.6 and 2.7). A brief comparison of this theorem and the inequality (5.2) in Theorem 5.1 leads to the following conclusions:

1. Theorem 6.3 covers both the definite case and the indefinite case, while the inequality (5.2) in Theorem 5.1 covers the definite case only;

2. When applying to the definite case, (5.2) is sharper than (6.6). As a matter of fact, (6.6) is a corollary of (5.2). It follows from (5.2) and Proposition 2.12 that if A is nonnegative definite

$$\begin{aligned}\sqrt{\sum_{i=1}^n \left[\text{RelDist}_2(\lambda_i, \widetilde{\lambda}_i)\right]^2} &\leq \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^n \left[\widetilde{\text{RelDist}}(\lambda_i, \widetilde{\lambda}_i)\right]^2} \\ &\leq \frac{1}{\sqrt{2}} \|\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}\|_F \\ &\leq \sqrt{\|I - \Sigma_{\rm d}\|_F^2 + \|I - \Sigma_{\rm d}^{-1}\|_F^2},\end{aligned}$$

   by Lemma 6.1 below.

Lemma 6.1

$$\frac{1}{\sqrt{2}} \|\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}\|_F \leq \sqrt{\|I - \Sigma_{\rm d}\|_F^2 + \|I - \Sigma_{\rm d}^{-1}\|_F^2},$$

and the equality holds if and only if $\Sigma_{\rm d} = I$, i.e., D is unitary.

Proof: Notice that for $\xi \in \mathbb{R}$

$$\left|\xi - \frac{1}{\xi}\right| \le \left|\xi - 1 + 1 - \frac{1}{\xi}\right| \le \sqrt{2}\sqrt{|\xi - 1|^2 + \left|1 - \frac{1}{\xi}\right|^2}$$

and the equality sign holds if and only if $\xi = 1$. ∎

The theorem below is a generalization of Theorems 4.3 and 4.4 for the spectral norm and that of Theorem 4.5.

Theorem 6.4 *To the hypotheses of Theorem 6.1 adds this: all λ_i's and $\widetilde{\lambda}_j$'s are nonnegative and are arranged descendingly as described in (3.4). Then we have*

$$\begin{aligned} \max_{1\le i\le n} \mathrm{RelDist}_p(\lambda_i, \widetilde{\lambda}_i) \quad \le \quad & \kappa_r(X)\kappa_r(\widetilde{X}) \min\left\{ \sqrt[q]{\|I - D_1^*\|_r^q + \|I - D_2^{-1}\|_r^q}\,, \right. \\ & \left. \sqrt[q]{\|I - D_1^{-*}\|_r^q + \|I - D_2\|_r^q} \right\}, \end{aligned} \tag{6.7}$$

where $1 \le r \le \infty$.

Similarly to Theorem 6.1, there is a stronger version of this theorem as follows.

Theorem 6.4s *Let all conditions of Theorem 6.4 hold. Then*

$$\max_{1\le i\le n} \mathrm{RelDist}_p(\lambda_i, \widetilde{\lambda}_i) \le \kappa_r(X)\kappa_r(\widetilde{X}) \times \tag{6.8}$$

$$\min_{U\in\mathbb{U}_n} \min\left\{ \sqrt[q]{\|U - D_1\|_2^q + \|U^* - D_2^{-1}\|_2^q}, \sqrt[q]{\|U^* - D_1^{-1}\|_2^q + \|U - D_2\|_2^q} \right\}.$$

As a consequence of this theorem and the solution (6.5) to the optimization problem (6.4), we deduce that

Theorem 6.5 *Under the conditions of Theorem 6.3, if A is nonnegative definite and the eigenvalues of A and $\widetilde{A}$ are in descending order as in (3.4), then*

$$\max_{1\le i\le n} \mathrm{RelDist}_p(\lambda_i, \widetilde{\lambda}_i) = \sqrt[q]{\|I - \Sigma_{\mathrm{d}}\|_2^q + \|I - \Sigma_{\mathrm{d}}^{-1}\|_2^q}, \tag{6.9}$$

where Σ_{d} is defined in (5.3).

However, there is not much interest in this theorem for two reasons: One is that (6.9) works for nonnegative definite matrices only just like the inequality (5.1) of Theorem 5.1; and the other is that (6.9) is less sharper than (5.1). To see this, we notice that (5.1) and Proposition 2.12 imply that

$$\max_{1\le i\le n} \mathrm{RelDist}_p(\lambda_i, \widetilde{\lambda}_i) \le 2^{-1/p}\widetilde{\mathrm{RelDist}}(\lambda_i, \widetilde{\lambda}_i) \le 2^{-1/p}\|\Sigma_{\mathrm{d}} - \Sigma_{\mathrm{d}}^{-1}\|_2.$$

So with Lemma 6.2 below, one can deduce (6.9) from (5.1). But still (6.9) looks nice and clean.

Lemma 6.2

$$\|\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}\|_2 \le 2^{1/p} \sqrt[q]{\|I - \Sigma_{\rm d}\|_2^q + \|I - \Sigma_{\rm d}^{-1}\|_2^q}, \tag{6.10}$$

and the equality holds if and only if $\Sigma_{\rm d} = I$, i.e., D is unitary.

Proof: Let $\xi \in \sigma(D)$ so that $\|\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}\|_2 = \left|\xi - \frac{1}{\xi}\right|$. Then

$$\begin{aligned}
\|\Sigma_{\rm d} - \Sigma_{\rm d}^{-1}\|_2 &= \left|\xi - \frac{1}{\xi}\right| \le |\xi - 1| + \left|1 - \frac{1}{\xi}\right| \\
&\le 2^{1/p} \sqrt[q]{|\xi - 1|^q + \left|1 - \frac{1}{\xi}\right|^q} \\
&\le 2^{1/p} \sqrt[q]{\|I - \Sigma_{\rm d}\|_2^q + \|I - \Sigma_{\rm d}^{-1}\|_2^q},
\end{aligned}$$

as required. ∎

So far we have considered the case when both A and $\widetilde{A}$ are diagonalizable. In what follows, we weaken this assumption by requiring only A to be diagonalizable and derive relative eigenvalue perturbation bounds of Bauer-Fike Type [2].

Theorem 6.6 *Assume that $A \in \mathbb{C}^{n\times n}$ is diagonalizable and admits the following decomposition*

$$A = X\Lambda X^{-1} \quad \textit{where} \quad \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n). \tag{6.11}$$

Assume[2] also either $\widetilde{A} = DA$ or $\widetilde{A} = AD$. Then for any $\widetilde{\lambda} \in \lambda(\widetilde{A})$ there exists a $\lambda \in \lambda(A)$ such that

$$\min_{\lambda\in\lambda(A)} \frac{|\widetilde{\lambda} - \lambda|}{|\lambda|} \le \|X^{-1}(D - I)X\|_p \le \kappa_p(X)\|I - D\|_p. \tag{6.12}$$

6.2 Singular Value Variations

As to singular value variations, we will prove

Theorem 6.7 *Let B and $\widetilde{B} = D_1^* B D_2$ be two $m \times n$ matrices, where D_1 and D_2 are nonsingular. Denote their singular values as in (3.2). Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\begin{aligned}
&\sqrt{\sum_{i=1}^n \left[\mathrm{RelDist}_2(\sigma_i, \widetilde{\sigma}_{\tau(i)})\right]^2} \\
&\le \frac{1}{\sqrt{2}} \sqrt{\|I - D_1\|_F^2 + \|I - D_1^{-1}\|_F^2 + \|I - D_2\|_F^2 + \|I - D_2^{-1}\|_F^2}.
\end{aligned} \tag{6.13}$$

[2] Unlike in our previous theorems, here we do not have to assume that D is nonsingular. Of course, if D is far away from I, the bound (6.12) does not tell us much; if D is close enough to I, it has to be nonsingular.

For any given $U \in \mathbb{U}_m$ and $V \in \mathbb{U}_n$, $U\widetilde{B}V^* = (D_1 U^*)^* B D_2 V^*$ has the same singular values as $\widetilde{B}$ does. Let the SVDs of D_1 and D_2 be as

$$D_1 = U_{\rm d1}\Sigma_{\rm d1}V_{\rm d1}^* \quad \text{and} \quad D_2 = U_{\rm d2}\Sigma_{\rm d2}V_{\rm d2}^*. \tag{6.14}$$

Applying Theorem 6.7 to matrices B and $U\widetilde{B}V^*$, together with the solution (6.5) to the optimization problem (6.4), leads to the following stronger version of the theorem.

Theorem 6.7s *Let all conditions of Theorem 6.7 hold. Then there is a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\begin{aligned}
&\sqrt{\sum_{i=1}^n \left[\mathrm{RelDist}_2(\sigma_i, \widetilde{\sigma}_{\tau(i)})\right]^2} \\
&\le \frac{1}{\sqrt{2}} \min_{U\in\mathbb{U}_m,\, V\in\mathbb{U}_n} \sqrt{\|U - D_1\|_F^2 + \|U^* - D_1^{-1}\|_F^2 + \|V - D_2\|_F^2 + \|V^* - D_2^{-1}\|_F^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{\|I - \Sigma_{\rm d1}\|_F^2 + \|I - \Sigma_{\rm d1}^{-1}\|_F^2 + \|I - \Sigma_{\rm d2}\|_F^2 + \|I - \Sigma_{\rm d2}^{-1}\|_F^2},
\end{aligned} \tag{6.15}$$

where $\Sigma_{\rm d1}$ and $\Sigma_{\rm d2}$ are defined in (6.14).

Theorems 6.7 and 6.7s are of less interest since they provide less sharper bounds than Theorem 5.2 does. We keep them around for comparison purpose, though still they look pretty. Now, we are going to show how to derive (6.15) from (5.5) of Theorem 5.2. It follows from (5.5) and Proposition 2.12 that

$$\begin{aligned}
&\sqrt{\sum_{i=1}^n \left[\mathrm{RelDist}_2(\sigma_i, \widetilde{\sigma}_i)\right]^2} \le \frac{1}{\sqrt{2}}\sqrt{\sum_{i=1}^n \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} \\
&\le \frac{1}{2\sqrt{2}}\left(\|\Sigma_{\rm d1} - \Sigma_{\rm d1}^{-1}\|_F + \|\Sigma_{\rm d2} - \Sigma_{\rm d2}^{-1}\|_F\right) \\
&\le \frac{1}{2}\left(\sqrt{\|I - \Sigma_{\rm d1}\|_F^2 + \|I - \Sigma_{\rm d1}^{-1}\|_F^2} + \sqrt{\|I - \Sigma_{\rm d2}\|_F^2 + \|I - \Sigma_{\rm d2}^{-1}\|_F^2}\right) \\
&\qquad\qquad \text{(by Lemma 6.1)} \\
&\le \frac{1}{\sqrt{2}}\sqrt{\|I - \Sigma_{\rm d1}\|_F^2 + \|I - \Sigma_{\rm d1}^{-1}\|_F^2 + \|I - \Sigma_{\rm d2}\|_F^2 + \|I - \Sigma_{\rm d2}^{-1}\|_F^2},
\end{aligned}$$

which shows (6.15). The proof in §10 of Theorem 6.7 is, however, of different spirit.

Theorem 6.8 *Let B and $\widetilde{B} = D_1^* B D_2$ be two $m \times n$ matrices, where D_1 and D_2 are nonsingular. Denote their singular values as in (3.2), and arrange the singular values of B and $\widetilde{B}$ in descending order respectively as in (3.3). Then* *we have the following*

$$\max_{1\le i\le n} \mathrm{RelDist}_p(\sigma_i,\widetilde{\sigma}_i) \le \min\left\{ \sqrt[q]{\|I-D_1^{-1}\|_2^q+\|I-D_2\|_2^q}\,, \sqrt[q]{\|I-D_1\|_2^q+\|I-D_2^{-1}\|_2^q} \right\}. \quad (6.16)$$

Similarly, applying Theorem 6.8 to matrices B and $U\widetilde{B}V^*$, we will have

Theorem 6.8s *Let all conditions of Theorem 6.8 hold. Then*

$$\begin{aligned} &\max_{1\le i\le n} \mathrm{RelDist}_p(\sigma_i,\widetilde{\sigma}_i) \\ &\le \min_{U\in\mathbb{U}_m,\,V\in\mathbb{U}_n} \min\left\{ \sqrt[q]{\|U^*-D_1^{-1}\|_2^q+\|V-D_2\|_2^q},\ \sqrt[q]{\|U-D_1\|_2^q+\|V^*-D_2^{-1}\|_2^q} \right\} \\ &= \min\left\{ \sqrt[q]{\|I-\Sigma_{\mathrm{d1}}^{-1}\|_2^q+\|I-\Sigma_{\mathrm{d2}}\|_2^q},\ \sqrt[q]{\|I-\Sigma_{\mathrm{d1}}\|_2^q+\|I-\Sigma_{\mathrm{d2}}^{-1}\|_2^q} \right\}, \end{aligned} \quad (6.17)$$

where Σ_{d1} *and* Σ_{d2} *are defined in (6.14).*

We can not say for sure that (5.4) of Theorem 5.2 is always sharper than the inequality (6.17), but many evidences indicates so. Let's weaken (6.17) a little bit into

$$\begin{aligned} &\max_{1\le i\le n} \mathrm{RelDist}_p(\sigma_i,\widetilde{\sigma}_i) \\ &\le \frac{1}{2}\left(\sqrt[q]{\|I-\Sigma_{\mathrm{d1}}^{-1}\|_2^q+\|I-\Sigma_{\mathrm{d2}}\|_2^q} + \sqrt[q]{\|I-\Sigma_{\mathrm{d1}}\|_2^q+\|I-\Sigma_{\mathrm{d2}}^{-1}\|_2^q} \right) \end{aligned} \quad (6.18)$$

(6.18) degrades (6.17) marginally in interesting cases. In what follows we will show that (6.18) is a consequence of Theorem 5.2. To this end, let $\xi\in\sigma(D_1)$ and $\zeta\in\sigma(D_2)$ so that

$$\|D_1^*-D_1^{-1}\|_2 = \left|\xi-\frac{1}{\xi}\right| \quad \text{and} \quad \|D_2^*-D_2^{-1}\|_2 = \left|\zeta-\frac{1}{\zeta}\right|.$$

We notice that

$$\begin{aligned} &\mathrm{RelDist}_p(\sigma_i,\widetilde{\sigma}_i) \le 2^{-1/p}\widetilde{\mathrm{RelDist}}(\sigma_i,\widetilde{\sigma}_i) && \text{(by Proposition 2.12)} \\ &\le \frac{1}{2^{1+1/p}}\left(\|\Sigma_{\mathrm{d1}}-\Sigma_{\mathrm{d1}}^{-1}\|_2+\|\Sigma_{\mathrm{d2}}-\Sigma_{\mathrm{d2}}^{-1}\|_2\right) && \text{(by Theorem 5.2)} \\ &= \frac{1}{2^{1+1/p}}\left(\left|\xi-\frac{1}{\xi}\right|+\left|\zeta-\frac{1}{\zeta}\right|\right) \\ &\le \frac{1}{2^{1+1/p}}\left(|\xi-1|+\left|1-\frac{1}{\xi}\right|+|\zeta-1|+\left|1-\frac{1}{\zeta}\right|\right) \\ &\le \frac{1}{2}\left(\sqrt[q]{|\xi-1|^q+\left|1-\frac{1}{\zeta}\right|^q} + \sqrt[q]{\left|1-\frac{1}{\xi}\right|^q+|\zeta-1|^q} \right) \end{aligned}$$

$$\leq \frac{1}{2}\left(\sqrt[q]{\|I-\Sigma_{\mathrm{d}1}\|_2^q+\|I-\Sigma_{\mathrm{d}2}^{-1}\|_2^q}+\sqrt[q]{\|I-\Sigma_{\mathrm{d}1}^{-1}\|_2^q+\|I-\Sigma_{\mathrm{d}2}\|_2^q}\right),$$

which gives (6.18).

7 A Theorem of Ostrowski and Other Theorems

In this section, we briefly review the current state of research on the problems listed in §1.1, together with our remarks.

Let A be an $n \times n$ Hermitian matrix. Perturbing A to D^*AD, where D is nonsingular, is actually performing a congruence transformation to A by D. The following theorem is due to Ostrowski [17, pp. 224–225].

Theorem 7.1 (Ostrowski) *Let A, $D \in \mathbb{C}^{n\times n}$ with A Hermitian and D nonsingular. Define $\widetilde{A} = D^*AD$. Denote the eigenvalues of A and $\widetilde{A}$ as in (3.1) and arrange them in the order as specified by (3.4). Then there exist θ_j's so that*

$$\sigma_{\min}(D)^2 \le \theta_j \le \sigma_{\max}(D)^2 \quad \text{and} \quad \widetilde{\lambda}_j = \theta_j \lambda_j,$$

for $j = 1, 2, \cdots, n$.

Ostrowski theorem implies immediately a relative perturbation bound on Hermitian eigenvalues.

Theorem 7.2 *Let the conditions of Theorem 7.1 hold. Then*

$$\frac{|\widetilde{\lambda}_j - \lambda_j|}{|\lambda_j|} \le \|I - D^*D\|_2,$$

or in another words,

$$\widetilde{\lambda}_j = \lambda_j(1 + \delta_j) \quad \text{with} \quad |\delta_j| \le \|I - D^*D\|_2,$$

for $j = 1, 2, \cdots, n$.

Although the inequality (5.1) of Theorem 5.1 and Theorem 7.2 are independent in the sense that one can not be inferred from the other, the latter is practically more useful in the following aspects:

1. Theorem 7.2 covers more while the inequality (5.1) of Theorem 5.1 covers nonnegative definite matrices only;

2. Theorem 7.2 is more friendly in the sense that it bounds directly on δ_j in the expression $\widetilde{\lambda}_j = \lambda_j(1 + \delta_j)$ which makes it easy to bound variations of $\mathrm{RelDist}_p$ as shown in Proposition 2.3 and Part II of this series [22].

Ostrowski theorem also applies to singular value problems of matrices B and $\widetilde{B} = D_1^* B D$ by working with Hermitian matrices

$$\begin{pmatrix} & B^* \\ B & \end{pmatrix} \text{ and } \begin{pmatrix} & \widetilde{B}^* \\ \widetilde{B} & \end{pmatrix} = \begin{pmatrix} D_2 & \\ & D_1 \end{pmatrix}^* \begin{pmatrix} & B^* \\ B & \end{pmatrix} \begin{pmatrix} D_2 & \\ & D_1 \end{pmatrix}. \tag{7.1}$$

Corollary 7.1 *Let B and $\widetilde{B} = D_1^* B D_2$ be two $m \times n$ matrices, where D_1 and D_2 are nonsingular. Denote their singular values as in (3.2) and arrange them in descending order respectively as in (3.3). Then*

$$\min\{\sigma_{\min}(D_1)^2, \sigma_{\min}(D_2)^2\} \le \frac{\widetilde{\sigma}_j}{\sigma_j} \le \max\{\sigma_{\max}(D_1)^2, \sigma_{\max}(D_2)^2\}$$

which gives

$$\frac{|\widetilde{\sigma}_j - \sigma_j|}{\sigma_j} \le \max\{\|I - D_1^* D_1\|_2,\, \|I - D_2^* D_2\|_2\},$$

or in another words,

$$\widetilde{\sigma}_j = \sigma_j(1+\gamma_j) \quad \text{with} \quad |\gamma_j| \le \max\{\|I - D_1^* D_1\|_2,\, \|I - D_2^* D_2\|_2\}.$$

for $j = 1, 2, \cdots, n$.

This corollary, though it is an immediate consequence of the above Ostrowski theorem and the equation (7.1), has appeared no where. Corollary 7.1 also has a advantage over Theorem 6.8s and the inequality (5.4) of Theorem 5.2 in that it bounds directly on γ_j in the expression $\widetilde{\sigma}_j = \sigma_j(1+\gamma_j)$. Of course, one can develop bounds on γ_j with little effort from Theorem 6.8s and Theorem 5.2. It turns out that Corollary 7.1 provides a less sharper bound than the following theorem due to Eisenstat and Ipsen [10].

Theorem 7.3 (Eisenstat-Ipsen) *Assume the conditions are as described in Corollary 7.1. Then*

$$\sigma_{\min}(D_1)\sigma_{\min}(D_2) \le \frac{\widetilde{\sigma}_j}{\sigma_j} \le \sigma_{\max}(D_1)\sigma_{\max}(D_2)$$

which yields

$$\frac{|\widetilde{\sigma}_j - \sigma_j|}{\sigma_j} \le \max\{|1 - \sigma_{\min}(D_1)\sigma_{\min}(D_2)|,\, |1 - \sigma_{\max}(D_1)\sigma_{\max}(D_2)|\},$$

or in another words, $\widetilde{\sigma}_j = \sigma_j(1+\gamma_j)$ with

$$|\gamma_j| \le \max\{|1 - \sigma_{\min}(D_1)\sigma_{\min}(D_2)|,\, |1 - \sigma_{\max}(D_1)\sigma_{\max}(D_2)|\},$$

for $j = 1, 2, \cdots, n$.

Theorem 7.3 always provide a sharper bound than Corollary 7.1 does, as the following lemma indicates.

Lemma 7.1 *For $\xi, \zeta \ge 0$,*

$$\max\{|1-\xi^2|, |1-\zeta^2|\} \ge |1-\xi\zeta|, \tag{7.2}$$

and the equality sign holds if and only if $\xi = \zeta$.

Proof: The inequality is obvious if either $\max\{\xi,\zeta\} \le 1$ or $\min\{\xi,\zeta\} \ge 1$. It is also clear if either $\xi = 1$ or $\zeta = 1$. Now it suffices for us to consider the case when $0 \le \xi < 1 < \zeta$.

1. $1-\xi^2 \ge \zeta^2 - 1 \Rightarrow \xi^2+\zeta^2 \le 2 \Rightarrow \xi\zeta < 1 \Rightarrow 1-\xi^2 > 1-\xi\zeta = |1-\xi\zeta|$;
2. $1-\xi^2 < \zeta^2-1 \Rightarrow \xi^2+\zeta^2 > 2 \Rightarrow \xi\zeta+\zeta^2 \ge \xi^2+\zeta^2 > 2 \Rightarrow \zeta^2-1 > 1-\xi\zeta$; also $\zeta^2 > \xi\zeta \Rightarrow \zeta^2 - 1 > \xi\zeta - 1$. So $\zeta^2 - 1 > |1-\xi\zeta|$.

From the above proof, it is clear that $\max\{|1-\xi^2|, |1-\zeta^2|\} = |1-\xi\zeta|$ if and only if $\xi = \zeta$. ∎

Regarding to graded matrices, the following two theorems are due to Demmel & Veselić [9] and Mathias [25].

Theorem 7.4 (Demmel-Veselić) *Let the conditions of Theorem 5.4 hold. Arrange the eigenvalues of $H = D^*AD$ and $\widetilde{H} = D^*\widetilde{A}D$ descendingly as in (3.4). Then*

$$\frac{|\widetilde{\lambda}_j - \lambda_j|}{|\lambda_j|} \le \|A^{-1}\|_2 \|\Delta A\|_2$$

or in another words,

$$\widetilde{\lambda}_j = \lambda_j(1+\delta_j) \quad \text{with} \quad |\delta_j| \le \|A^{-1}\|_2\|\Delta A\|_2,$$

for $j = 1, 2, \cdots, n$.

Theorem 7.5 (Mathias) *Let the conditions of Theorem 5.3 hold. Arrange the singular values of $G = BD$ and $\widetilde{G} = \widetilde{B}D$ descendingly as in (3.3). Then*

$$\frac{|\widetilde{\sigma}_j - \sigma_j|}{\sigma_j} \le \|B^{-1}\|_2\|\Delta B\|_2,$$

or in another words,

$$\widetilde{\sigma}_j = \sigma_j(1+\gamma_j) \quad \text{with} \quad |\gamma_j| \le \|B^{-1}\|_2\|\Delta B\|_2,$$

for $j = 1, 2, \cdots, n$.

Finally, let us see what we can get from Theorems 7.2, 7.4, 7.5 and 7.3 and Corollary 7.1, in terms of the two kinds of relative distances defined in §2.

1. From Theorem 7.2, it follows

$$\begin{aligned} \mathrm{RelDist}_p(\lambda_j, \widetilde{\lambda}_j) \le \mathrm{RelDist}_\infty(\lambda_j, \widetilde{\lambda}_j) &\le \|I - D^*D\|_2, && (7.3)\\ \widetilde{\mathrm{RelDist}}(\lambda_j, \widetilde{\lambda}_j) &\le \frac{\|I-D^*D\|_2}{\sigma_{\min}(D)}. && (7.4) \end{aligned}$$

The inequality (7.3) holds because

$$\mathrm{RelDist}_\infty(\lambda_j, \widetilde{\lambda}_j) = \frac{|\widetilde{\lambda}_j - \lambda_j|}{\max\{|\lambda_j|, |\widetilde{\lambda}_j|\}} \le \frac{|\widetilde{\lambda}_j - \lambda_j|}{|\lambda_j|} \le \|I - D^* D\|_2;$$

and the inequality (7.4) holds because

$$\widetilde{\mathrm{RelDist}}(\lambda_j, \widetilde{\lambda}_j) = \frac{|\widetilde{\lambda}_j - \lambda_j|}{\sqrt{|\lambda_j|\,|\widetilde{\lambda}_j|}} = \frac{|\widetilde{\lambda}_j - \lambda_j|}{|\lambda_j|}\sqrt{\frac{|\lambda_j|}{|\widetilde{\lambda}_j|}} \le \frac{\|I - D^* D\|_2}{\sigma_{\min}(D)}.$$

2. From Corollary 7.1, we have

$$\begin{aligned} \mathrm{RelDist}_\infty(\sigma_j, \widetilde{\sigma}_j) &\le \max\{\|I - D_1^* D_1\|_2,\, \|I - D_2^* D_2\|_2\}, && (7.5) \\ \widetilde{\mathrm{RelDist}}(\sigma_j, \widetilde{\sigma}_j) &\le \frac{\max\{\|I - D_1^* D_1\|_2,\, \|I - D_2^* D_2\|_2\}}{\min\{\sigma_{\min}(D_1),\, \sigma_{\min}(D_2)\}}. && (7.6) \end{aligned}$$

3. From Theorem 7.3, it follows

$$\begin{aligned} &\mathrm{RelDist}_\infty(\sigma_j, \widetilde{\sigma}_j) \\ &\le \max\{|1 - \sigma_{\min}(D_1)\sigma_{\min}(D_2)|,\, |1 - \sigma_{\max}(D_1)\sigma_{\max}(D_2)|\}, \quad (7.7) \\ &\widetilde{\mathrm{RelDist}}(\sigma_j, \widetilde{\sigma}_j) \\ &\le \frac{\max\{|1 - \sigma_{\min}(D_1)\sigma_{\min}(D_2)|,\, |1 - \sigma_{\max}(D_1)\sigma_{\max}(D_2)|\}}{\sqrt{\sigma_{\min}(D_1)\sigma_{\min}(D_2)}}. \quad (7.8) \end{aligned}$$

The inequalities (7.7) and (7.8) are sharper than (7.5) and (7.6), respectively.

4. From Theorem 7.4, we have

$$\begin{aligned} \mathrm{RelDist}_\infty(\lambda_j, \widetilde{\lambda}_j) &\le \|A^{-1}\|_2 \|\Delta A\|_2, && (7.9) \\ \widetilde{\mathrm{RelDist}}(\lambda_j, \widetilde{\lambda}_j) &\le \frac{\|A^{-1}\|_2 \|\Delta A\|_2}{\sqrt{1 - \|A^{-1}\|_2 \|\Delta A\|_2}}. && (7.10) \end{aligned}$$

The inequality (7.10) has been derived in Theorem 5.4.

5. From Theorem 7.5, it follows

$$\begin{aligned} \mathrm{RelDist}_\infty(\sigma_j, \widetilde{\sigma}_j) &\le \|B^{-1}\|_2 \|\Delta B\|_2, && (7.11) \\ \widetilde{\mathrm{RelDist}}(\sigma_j, \widetilde{\sigma}_j) &\le \frac{\|B^{-1}\|_2 \|\Delta B\|_2}{\sqrt{1 - \|B^{-1}\|_2 \|\Delta B\|_2}}. && (7.12) \end{aligned}$$

The inequality (7.12) turns out to be sharper than the last "$\le$" in (5.9) of Theorem 5.3.

8 Remarks on Generalized Eigenvalue Problems and Generalized Singular Value Problems

In this section, we are going to say a few words for the following perturbations. As we shall see, the results in previous sections, as well as those in Li [22], can be applied to derive relative perturbation bounds for them.

- **Generalized eigenvalue problem:**
  $H_1 - \lambda H_2 \equiv D_1^* A_1 D_1 - \lambda D_2^* A_2 D_2$ and $\widetilde{H}_1 - \lambda \widetilde{H}_2 \equiv D_1^* \widetilde{A}_1 D_1 - \lambda D_2^* \widetilde{A}_2 D_2$ with all A_i and $\widetilde{A}_i$ positive definite and $\|A_i^{-1}\|_2 \|\widetilde{A}_i - A_i\|_2 < 1$, where D_i are some square matrices and one of them are nonsingular.

- **Generalized singular problem:**
  $\{G_1, G_2\} \equiv \{B_1 D_1, B_2 D_2\}$ and $\{\widetilde{G}_1, \widetilde{G}_2\} \equiv \{\widetilde{B}_1 D_1, \widetilde{B}_2 D_2\}$ with all B_i and $\widetilde{B}_i$ nonsingular and $\|B_i^{-1}\|_2 \|\widetilde{B}_i - B_i\|_2 < 1$, where D_i are some square matrices and one of them is nonsingular.

For the above mention generalized eigenvalue problem, without loss of any generality, consider only the case when D_2 is nonsingular. Then the generalized eigenvalue problem for $H_1 - \lambda H_2 \equiv D_1^* A_1 D_1 - \lambda D_2^* A_2 D_2$ is equivalent to the standard eigenvalue problem for

$$A_2^{-1/2} D_2^{-1} D_1^* A_1 D_1 D_2^{-1} A_2^{-1/2}; \tag{8.1}$$

and the generalized eigenvalue problem for $\widetilde{H}_1 - \lambda \widetilde{H}_2 \equiv D_1^* \widetilde{A}_1 D_1 - \lambda D_2^* \widetilde{A}_2 D_2$ is equivalent to the standard eigenvalue problem for

$$\widehat{D}^* A_2^{-1/2} D_2^{-1} D_1^* \widetilde{A}_1 D_1 D_2^{-1} A_2^{-1/2} \widehat{D}, \tag{8.2}$$

where $\Delta A_2 \stackrel{\text{def}}{=} \widetilde{A}_2 - A_2$ and $\widehat{D} = \widehat{D}^* \stackrel{\text{def}}{=} (I + A_2^{-1/2} (\Delta A_2) A_2^{-1/2})^{-1/2}$. So bounding relative distances between the eigenvalues of $H_1 - \lambda H_2$ and these of $\widetilde{H}_1 - \lambda \widetilde{H}_2$ is transformed to bounding relative distances between the eigenvalues of the matrix (8.1) and these of the matrix (8.2). The latter can be accomplished in two steps:

1. Bounding relative distances between the eigenvalues of the matrix (8.1) and these of
$$\widehat{D}^* A_2^{-1/2} D_2^{-1} D_1^* A_1 D_1 D_2^{-1} A_2^{-1/2} \widehat{D}; \tag{8.3}$$
2. Bounding relative distances between the eigenvalues of the matrix (8.3) and these of the matrix (8.2).

As to the above mention generalized singular problem, we shall consider their corresponding generalized eigenvalue problems [20, 32, 34] for

$$D_1^* B_1^* B_1 D_1 - \lambda D_2^* B_2^* B_2 D_2 \quad \text{and} \quad D_1^* \widetilde{B}_1^* \widetilde{B}_1 D_1 - \lambda D_2^* \widetilde{B}_2^* \widetilde{B}_2 D_2,$$

instead.

9 Proofs of Theorems 6.1 and 6.4

To prove the theorems, we need a little preparation. A matrix $Y = (y_{ij}) \in \mathbb{R}^{n\times n}$ is *doubly stochastic* if all $y_{ij} \geq 0$ and

$$\sum_{k=1}^{n} y_{ik} = \sum_{k=1}^{n} y_{kj} = 1 \quad \text{for } k = 1, 2, \cdots, n.$$

A matrix $P \in \mathbb{R}^{n\times n}$ is called a *permutation matrix* if exactly one entry in each row and each column equals to 1 and all others are zero. Let e_i be the ith column vector of I_n. Each permutation matrix P corresponds to a unique permutation τ of $\{1, 2, \cdots, n\}$ so that

$$P = (e_{\tau(1)}, e_{\tau(2)}, \cdots, e_{\tau(n)}),$$

and vice versa. The following wonderful result is due to Birkhoff [5] (see also [17, pp. 527–528]).

Lemma 9.1 (Birkhoff) *An $n \times n$ matrix is doubly stochastic if and only if it lies in the convex hull of $n!$ permutation matrices.*

Lemma 9.2 *Let $Y = (y_{ij})$ be an $n \times n$ doubly stochastic matrix, and let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$. Then there exists a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\sum_{i,j=1}^{n} |m_{ij}|^2 y_{ij} \geq \sum_{i=1}^{n} |m_{i\tau(i)}|^2.$$

Proof: Denote all $n \times n$ permutation matrices as P_k, and their corresponding permutations of $\{1, 2, \cdots, n\}$ as τ_k, where $k = 1, 2, \cdots, n!$. It follows from Lemma 9.1 that Y can be written as

$$Y = \sum_{k=1}^{n!} \alpha_k P_k,$$

where $\alpha_k \geq 0$ and $\sum_{k=1}^{n!} \alpha_k = 1$. Hence

$$\sum_{i,j=1}^{n} |m_{ij}|^2 y_{ij} = \sum_{k=1}^{n!} \alpha_k \sum_{i=1}^{n} |m_{i\tau_k(i)}|^2 \geq \min_{1\leq k\leq n!} \sum_{i=1}^{n} |m_{i\tau_k(i)}|^2,$$

as was to be shown. ∎

The trick in the above proof is quite standard. It was first used by Hoffman and Wielandt [16], and Sun [31] used it to prove a Hoffman-Wielandt type theorem for a special class of matrix pencils.

The following lemma is due to Elsner and Friedland [12].

Lemma 9.3 (Elsner-Friedland) *Let $Y = (y_{ij}) \in \mathbb{C}^{n\times n}$. Then there exist two $n \times n$ doubly stochastic matrices Y_1, Y_2, so that entrywisely*

$$\sigma_{\min}(Y)^2\, Y_1 \le (|y_{ij}|^2) \le \sigma_{\max}(Y)^2\, Y_2,$$

where $\sigma_{\min}(Y)$ and $\sigma_{\max}(Y)$ are the smallest and largest singular values of Y, respectively.

Proof of Theorem 6.1: Let us first derive our perturbation equations.

$$\begin{aligned}
X^{-1}(A-\widetilde{A})\widetilde{X} &= \Lambda X^{-1}\widetilde{X} - X^{-1}\widetilde{X}\widetilde{\Lambda}, \\
A-\widetilde{A} &= A - D_1^* A D_2 = A - AD_2 + AD_2 - D_1^* A D_2 \\
&= A(I-D_2) + (D_1^{-*} - I)\widetilde{A}, \\
\widetilde{X}^{-1}(A-\widetilde{A})X &= \widetilde{X}^{-1}X\Lambda - \widetilde{\Lambda}\widetilde{X}^{-1}X, \\
A-\widetilde{A} &= A - D_1^* A D_2 = A - D_1^* A + D_1^* A - D_1^* A D_2 \\
&= (I-D_1^*)A + \widetilde{A}(D_2^{-1} - I).
\end{aligned}$$

Thus, we have

$$\Lambda X^{-1}\widetilde{X} - X^{-1}\widetilde{X}\widetilde{\Lambda} = \Lambda X^{-1}(I-D_2)\widetilde{X} + X^{-1}(D_1^{-*} - I)\widetilde{X}\widetilde{\Lambda}, \quad (9.1)$$

$$\widetilde{X}^{-1}X\Lambda - \widetilde{\Lambda}\widetilde{X}^{-1}X = \widetilde{X}^{-1}(I-D_1^*)X\Lambda + \widetilde{\Lambda}\widetilde{X}^{-1}(D_2^{-1} - I)X. \quad (9.2)$$

Set $Y \overset{\text{def}}{=} X^{-1}\widetilde{X} = (y_{ij})$, $E = X^{-1}(I-D_2)\widetilde{X} = (e_{ij})$ and $\widetilde{E} = X^{-1}(D_1^{-*} - I)\widetilde{X} = (\widetilde{e}_{ij})$. Then the equation (9.1) reads $\Lambda Y - Y\widetilde{\Lambda} = \Lambda E + \widetilde{E}\widetilde{\Lambda}$, or componentwisely $\lambda_i y_{ij} - y_{ij}\widetilde{\lambda}_j = \lambda_i e_{ij} + \widetilde{e}_{ij}\widetilde{\lambda}_j$, so

$$|(\lambda_i - \widetilde{\lambda}_j)y_{ij}|^2 \le (|\lambda_i|^2 + |\widetilde{\lambda}_j|^2)(|e_{ij}|^2 + |\widetilde{e}_{ij}|^2),$$

which yields

$$|e_{ij}|^2 + |\widetilde{e}_{ij}|^2 \ge \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_j)\right]^2 |y_{ij}|^2.$$

Hence

$$\|X^{-1}(I-D_2)\widetilde{X}\|_F^2 + \|X^{-1}(D_1^{-*} - I)\widetilde{X}\|_F^2 \ge \sum_{i,j=1}^n \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_j)\right]^2 |y_{ij}|^2 \quad (9.3)$$

which, together with Lemmas 9.3 and 9.2, show that

$$\|X^{-1}(I-D_2)\widetilde{X}\|_F^2 + \|X^{-1}(D_1^{-*} - I)\widetilde{X}\|_F^2 \ge \sigma_{\min}(Y)^2 \sum_{i=1}^n \left[\mathrm{RelDist}_2(\lambda_i, \widetilde{\lambda}_{\tau(i)})\right]^2$$

for some permutation τ of $\{1, 2, \cdots, n\}$. Since

$$\sigma_{\min}(Y) = \|Y^{-1}\|_2^{-1} = \|\widetilde{X}^{-1}X\|_2^{-1} \ge \|\widetilde{X}^{-1}\|_2^{-1}\|X\|_2^{-1},$$

so

$$
\begin{aligned}
&\|\widetilde{X}^{-1}\|_2\|X\|_2\sqrt{\|X^{-1}(I-D_2)\widetilde{X}\|_F^2+\|X^{-1}(D_1^{-*}-I)\widetilde{X}\|_F^2} \\
&\quad\geq \|\widetilde{X}^{-1}\|_2\|X\|_2\sigma_{\min}(Y)\sqrt{\sum_{i=1}^n\left[\mathrm{RelDist}_2(\lambda_i,\widetilde{\lambda}_{\tau(i)})\right]^2} \\
&\quad\geq \sqrt{\sum_{i=1}^n\left[\mathrm{RelDist}_2(\lambda_i,\widetilde{\lambda}_{\tau(i)})\right]^2}. \qquad (9.4)
\end{aligned}
$$

Set $\widetilde{Y}\stackrel{\mathrm{def}}{=}\widetilde{X}^{-1}X=(\widetilde{y}_{ij})$. Similarly, we get

$$
\|\widetilde{X}^{-1}(I-D_1^*)X\|_F^2+\|\widetilde{X}^{-1}(D_2^{-1}-I)X\|_F^2\geq\sum_{i,j=1}^n\left[\mathrm{RelDist}_2(\lambda_i,\widetilde{\lambda}_j)\right]^2|\widetilde{y}_{ji}|^2
$$

which, together with Lemmas 9.3 and 9.2, show that

$$
\|\widetilde{X}^{-1}(I-D_1^*)X\|_F^2+\|\widetilde{X}^{-1}(D_2^{-1}-I)X\|_F^2\geq\sigma_{\min}(\widetilde{Y})^2\sum_{i=1}^n\left[\mathrm{RelDist}_2(\lambda_i,\widetilde{\lambda}_{\tau(i)})\right]^2.
$$

Since

$$
\sigma_{\min}(\widetilde{Y})=\|\widetilde{Y}^{-1}\|_2^{-1}=\|X^{-1}\widetilde{X}\|_2^{-1}\geq\|X^{-1}\|_2^{-1}\|\widetilde{X}\|_2^{-1}.
$$

Along the lines as we were proceeding in (9.4), we will reach

$$
\begin{aligned}
&\|X^{-1}\|_2\|\widetilde{X}\|_2\sqrt{\|\widetilde{X}^{-1}(I-D_1^*)X\|_F^2+\|\widetilde{X}^{-1}(D_2^{-1}-I)X\|_F^2} \\
&\qquad\geq \sqrt{\sum_{i=1}^n\left[\mathrm{RelDist}_2(\lambda_i,\widetilde{\lambda}_{\tau(i)})\right]^2}. \qquad (9.5)
\end{aligned}
$$

The inequality (6.1) is now a simple consequence of (9.4) and (9.5). ∎

A proof of Theorem 6.4 is based on the following result due to Li [21, pp. 207–208]. For a $X\in\mathbb{C}^{m\times n}$, introduce the following notation for a $k\times\ell$ submatrix of $X=(x_{ij})$:

$$
X\begin{pmatrix} i_1\cdots i_k \\ j_1\cdots j_\ell\end{pmatrix}\stackrel{\mathrm{def}}{=}\begin{pmatrix} x_{i_1j_1} & x_{i_1j_2} & \cdots & x_{i_1j_\ell} \\ x_{i_2j_1} & x_{i_2j_2} & \cdots & x_{i_2j_\ell} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_kj_1} & x_{i_kj_2} & \cdots & x_{i_kj_\ell}\end{pmatrix}, \qquad (9.6)
$$

where $1\leq i_1<\cdots<i_k\leq n$ and $1\leq j_1<\cdots<j_\ell\leq n$.

Lemma 9.4 (Li) *Suppose that $X \in \mathbb{C}^{n\times n}$ is nonsingular, $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_\ell \le n$, and $k+\ell > n$. Then*

$$\left\| X \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_\ell \end{pmatrix} \right\|_p \ge \|X^{-1}\|_p^{-1}.$$

Moreover, if X is unitary then

$$\left\| X \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_\ell \end{pmatrix} \right\|_2 = 1.$$

Proof of Theorem 6.4: Let k be the index such that

$$\eta_p \stackrel{\text{def}}{=} \max_{1\le i\le n} \text{RelDist}_p(\lambda_i, \widetilde{\lambda}_i) = \text{RelDist}_p(\lambda_k, \widetilde{\lambda}_k).$$

If $\eta_p = 0$, the inequality (6.7) is trivial. Assume $\eta_p > 0$. Also assume, without lose of any generality, that

$$\lambda_k > \widetilde{\lambda}_k \ge 0.$$

Partition X, X^{-1}, $\widetilde{X}$ and $\widetilde{X}^{-1}$ as follows:

$$X = (X_1, X_2),\ X^{-1} = \begin{pmatrix} W_1^* \\ W_2^* \end{pmatrix},\ \widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2),\ \widetilde{X}^{-1} = \begin{pmatrix} \widetilde{W}_1^* \\ \widetilde{W}_2^* \end{pmatrix},$$

where X_1, $W_1 \in \mathbb{C}^{n\times k}$ and $\widetilde{X}_1$, $\widetilde{W}_1 \in \mathbb{C}^{n\times(k-1)}$, and write $\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$ and $\widetilde{\Lambda} = \text{diag}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2)$, where $\Lambda_1 \in \mathbb{R}^{k\times k}$ and $\widetilde{\Lambda}_1 \in \mathbb{R}^{(k-1)\times(k-1)}$. It follows from the equations (9.1) and (9.2) that

$$\begin{aligned} \Lambda_1 W_1^* \widetilde{X}_2 - W_1^* \widetilde{X}_2 \widetilde{\Lambda}_2 &= \Lambda_1 W_1^*(I - D_2)\widetilde{X}_2 + W_1^*(D_1^{-*} - I)\widetilde{X}_2\widetilde{\Lambda}_2, \quad (9.7) \\ \widetilde{W}_2^* X_1 \Lambda_1 - \widetilde{\Lambda}_2 \widetilde{W}_2^* X_1 &= \widetilde{W}_2^*(I - D_1^*)X_1\Lambda_1 + \widetilde{\Lambda}_2\widetilde{W}_2^*(D_2^{-1} - I)X_1 \quad (9.8) \end{aligned}$$

which gives

$$\begin{aligned} W_1^* \widetilde{X}_2 - \Lambda_1^{-1} W_1^* \widetilde{X}_2 \widetilde{\Lambda}_2 &= W_1^*(I - D_2)\widetilde{X}_2 + \Lambda_1^{-1} W_1^*(D_1^{-*} - I)\widetilde{X}_2\widetilde{\Lambda}_2, \quad (9.9) \\ \widetilde{W}_2^* X_1 - \widetilde{\Lambda}_2 \widetilde{W}_2^* X_1 \Lambda_1^{-1} &= \widetilde{W}_2^*(I - D_1^*)X_1 + \widetilde{\Lambda}_2\widetilde{W}_2^*(D_2^{-1} - I)X_1\Lambda_1^{-1}. \quad (9.10) \end{aligned}$$

Lemma 9.4 implies

$$\begin{aligned} \left\| W_1^* \widetilde{X}_2 \right\|_r &\ge \left\| (X^{-1}\widetilde{X})^{-1} \right\|_r^{-1} \ge \left\| \widetilde{X}^{-1} X \right\|_r^{-1} \ge \|\widetilde{X}^{-1}\|_r^{-1} \|X\|_r^{-1}, \\ \left\| \widetilde{W}_2^* X_1 \right\|_r &\ge \left\| (\widetilde{X}^{-1} X)^{-1} \right\|_r^{-1} \ge \left\| X^{-1} \widetilde{X} \right\|_r^{-1} \ge \|X^{-1}\|_r^{-1} \|\widetilde{X}\|_r^{-1}, \end{aligned}$$

since $W_1^* \widetilde{X}_2$ is a $k\times(n-k+1)$ submatrix of $X^{-1}\widetilde{X}$, and $\widetilde{W}_2^* X_1$ is a $(n-k+1)\times k$ submatrix of $\widetilde{X}^{-1}X$ and $k + (n-k+1) = n+1 > n$. So it follows from (9.9) that

$$
\begin{aligned}
&\left(1-\frac{\widetilde{\lambda}_k}{\lambda_k}\right)\|\widetilde{X}^{-1}\|_r^{-1}\|X\|_r^{-1}\\
&\le \quad \left(1-\frac{\widetilde{\lambda}_k}{\lambda_k}\right)\left\|W_1^*\widetilde{X}_2\right\|_r\\
&= \quad \left\|W_1^*\widetilde{X}_2\right\|_r - \|\Lambda_1^{-1}\|_r\left\|W_1^*\widetilde{X}_2\right\|_r\left\|\widetilde{\Lambda}_2\right\|_r\\
&\le \quad \left\|W_1^*\widetilde{X}_2\right\|_r - \left\|\Lambda_1^{-1}W_1^*\widetilde{X}_2\widetilde{\Lambda}_2\right\|_r\\
&\le \quad \left\|W_1^*\widetilde{X}_2 - \Lambda_1^{-1}W_1^*\widetilde{X}_2\widetilde{\Lambda}_2\right\|_r\\
&= \quad \left\|W_1^*(I-D_2)\widetilde{X}_2 + \Lambda_1^{-1}W_1^*(D_1^{-*}-I)\widetilde{X}_2\widetilde{\Lambda}_2\right\|_r\\
&\le \quad \left\|W_1^*(I-D_2)\widetilde{X}_2\right\|_r + \frac{\widetilde{\lambda}_k}{\lambda_k}\left\|W_1^*(D_1^{-*}-I)\widetilde{X}_2\right\|_r\\
&\le \quad \|W_1^*\|_r\|\widetilde{X}_2\|_r\left(\|I-D_2\|_r + \frac{\widetilde{\lambda}_k}{\lambda_k}\|D_1^{-*}-I\|_r\right)\\
&\le \quad \|X^{-1}\|_r\|\widetilde{X}\|_r\sqrt[p]{1+\frac{\widetilde{\lambda}_k^p}{\lambda_k^p}}\sqrt[q]{\|I-D_2\|_r^q + \|I-D_1^{-*}\|_r^q}.
\end{aligned}
$$

Similarly, it follows from (9.10) that

$$
\begin{aligned}
&\left(1-\frac{\widetilde{\lambda}_k}{\lambda_k}\right)\|X^{-1}\|_r^{-1}\|\widetilde{X}\|_r^{-1}\\
&\le \quad \|\widetilde{X}^{-1}\|_r\|X\|_r\sqrt[p]{1+\frac{\widetilde{\lambda}_k^p}{\lambda_k^p}}\sqrt[q]{\|I-D_2^{-1}\|_r^q + \|I-D_1^*\|_r^q}.
\end{aligned}
$$

The inequality (6.7) is now a simple consequence of above inequalities. ∎

10 Proofs of Theorems 6.7 and 6.8

Proof of Theorem 6.7: We assume, without lose of any generality, that $m = n$; otherwise, we can augment B and $\widetilde{B}$ with zero blocks of suitable size. For example if $m > n$, we do

$$B_1 = (B, 0_{m,m-n}), \quad \widetilde{B}_1 = (\widetilde{B}, 0_{m,m-n}) = D_1^* B_1 \mathrm{diag}(D_2, I_{m-n}).$$

Since this way only increases the number of zero singular values, and Proposition 2.7 says that zero singular values should be always paired to zero ones, we still have (6.13) in the end once we prove it for B_1 and $\widetilde{B}_1$.

Assume now $m = n$ and let the singular value decompositions of B and $\widetilde{B}$ be as

$$B = U\Sigma V^* \quad \text{and} \quad \widetilde{B} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*, \tag{10.1}$$

where U, V, $\widetilde{U}$, $\widetilde{V} \in \mathbb{U}_n$ and

$$\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_n) \quad \text{and} \quad \widetilde{\Sigma} = \mathrm{diag}(\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_n). \tag{10.2}$$

Notice

$$\begin{aligned} U^*(B - \widetilde{B})\widetilde{V} &= \Sigma V^* \widetilde{V} - U^* \widetilde{U}\widetilde{\Sigma}, \\ B - \widetilde{B} &= B - D_1^* B D_2 = B - BD_2 + BD_2 - D_1^* B D_2 \\ &= B(I - D_2) + (D_1^{-*} - I)\widetilde{B}. \end{aligned}$$

Thus, we have

$$\Sigma V^* \widetilde{V} - U^* \widetilde{U}\widetilde{\Sigma} = \Sigma V^*(I - D_2)\widetilde{V} + U^*(D_1^{-*} - I)\widetilde{U}\widetilde{\Sigma}. \tag{10.3}$$

One the other hand, we have

$$\begin{aligned} \widetilde{U}^*(B - \widetilde{B})V &= \widetilde{U}^* U\Sigma - \widetilde{\Sigma}\widetilde{V}^* V, \\ B - \widetilde{B} &= B - D_1^* B D_2 = B - D_1^* B + D_1^* B - D_1^* B D_2 \\ &= (I - D_1^*)B + \widetilde{B}(D_2^{-1} - I). \end{aligned}$$

Thus, we have

$$\widetilde{U}^* U\Sigma - \widetilde{\Sigma}\widetilde{V}^* V = \widetilde{U}^*(I - D_1^*)U\Sigma + \widetilde{\Sigma}\widetilde{V}^*(D_2^{-1} - I)V.$$

Taking conjugate transpose in both sides, we get

$$\Sigma U^* \widetilde{U} - V^* \widetilde{V}\widetilde{\Sigma} = \Sigma U^*(I - D_1)\widetilde{U} + V^*(D_2^{-*} - I)\widetilde{V}\widetilde{\Sigma}. \tag{10.4}$$

Set $Q = U^*\widetilde{U} = (q_{ij})$ and $\widetilde{Q} = V^*\widetilde{V} = (\widetilde{q}_{ij})$. Both are unitary. Similarly to the derivation of the inequality (9.3), from the perturbation equations (10.3) and (10.4) one can get

$$\|I - D_2\|_F^2 + \|I - D_1^{-*}\|_F^2 \geq \sum_{i,j=1}^{n} \frac{|\sigma_i \widetilde{q}_{ij} - q_{ij}\widetilde{\sigma}_j|^2}{\sigma_i^2 + \widetilde{\sigma}_j^2}, \tag{10.5}$$

$$\|I - D_1\|_F^2 + \|I - D_2^{-*}\|_F^2 \geq \sum_{i,j=1}^{n} \frac{|\sigma_i q_{ij} - \widetilde{q}_{ij}\widetilde{\sigma}_j|^2}{\sigma_i^2 + \widetilde{\sigma}_j^2}. \tag{10.6}$$

Since

$$\begin{aligned} |\sigma_i \widetilde{q}_{ij} - q_{ij}\widetilde{\sigma}_j|^2 + |\sigma_i q_{ij} - \widetilde{q}_{ij}\widetilde{\sigma}_j|^2 &= \sigma_i^2 |\widetilde{q}_{ij}|^2 + |q_{ij}|^2 \widetilde{\sigma}_j^2 - 2\Re(\sigma_i \widetilde{q}_{ij} \bar{q}_{ij} \widetilde{\sigma}_j) \\ &\quad + \sigma_i^2 |q_{ij}|^2 + |\widetilde{q}_{ij}|^2 \widetilde{\sigma}_j^2 - 2\Re(\sigma_i \bar{q}_{ij} \widetilde{q}_{ij} \widetilde{\sigma}_j) \\ &\geq (\sigma_i - \widetilde{\sigma}_j)^2 (|q_{ij}|^2 + |\widetilde{q}_{ij}|^2), \end{aligned}$$

where $\Re(\cdot)$ takes the real part of a complex number. The last "$\geq$" holds because

$$\begin{aligned} 2\Re(\sigma_i \widetilde{q}_{ij} \bar{q}_{ij} \widetilde{\sigma}_j) &\leq \sigma_i \widetilde{\sigma}_j (|q_{ij}|^2 + |\widetilde{q}_{ij}|^2), \\ 2\Re(\sigma_i \bar{q}_{ij} \widetilde{q}_{ij} \widetilde{\sigma}_j) &\leq \sigma_i \widetilde{\sigma}_j (|q_{ij}|^2 + |\widetilde{q}_{ij}|^2). \end{aligned}$$

Now adding the corresponding two sides of the inequalities (10.5) and (10.6) leads to

$$\begin{aligned} &\|I - D_2\|_F^2 + \|I - D_1^{-*}\|_F^2 + \|I - D_1\|_F^2 + \|I - D_2^{-*}\|_F^2 \\ &\geq 2 \sum_{i,j=1}^{n} [\mathrm{RelDist}_2(\sigma_i, \widetilde{\sigma}_j)]^2 \, \frac{|q_{ij}|^2 + |\widetilde{q}_{ij}|^2}{2}. \end{aligned}$$

It is easy to see that the matrix whose (i,j)th entry is $\frac{|q_{ij}|^2 + |\widetilde{q}_{ij}|^2}{2}$ is a doubly stochastic matrix. Hence applying Lemma 9.2 leads to the inequality (6.13). ∎

Proof of Theorem 6.8: Similarly to the remark we made at the beginning of the above proof, we may assume, without lose of any generality, that $m = n$ because of Proposition 2.5. Then still, we have the perturbation equations (10.3) and (10.4). Let k be the index such that

$$\eta_p \stackrel{\mathrm{def}}{=} \max_{1 \leq i \leq n} \mathrm{RelDist}_p(\sigma_i, \widetilde{\sigma}_i) = \mathrm{RelDist}_p(\sigma_k, \widetilde{\sigma}_k).$$

If $\eta_p = 0$, the inequality (6.16) is trivial. Assume $\eta_p > 0$. Also assume, without lose of generality, that

$$\sigma_k > \widetilde{\sigma}_k \geq 0.$$

Partition U, V, $\widetilde{U}$, $\widetilde{V}$ as follows

$$U = (U_1, U_2),\ V = (V_1, V_2),\ \widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2) \quad \text{and} \quad \widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2),$$

where $U_1, V_1 \in \mathbb{C}^{n\times k}$ and $\widetilde{U}_1, \widetilde{V}_1 \in \mathbb{C}^{n\times(k-1)}$. Write $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2)$ and $\widetilde{\Sigma} = \mathrm{diag}(\widetilde{\Sigma}_1, \widetilde{\Sigma}_2)$, where $\Sigma_1 \in \mathbb{R}^{k\times k}$ and $\widetilde{\Sigma}_1 \in \mathbb{R}^{(k-1)\times(k-1)}$. It follows from the equations (10.3) and (10.4) that

$$\begin{aligned} \Sigma_1 V_1^* \widetilde{V}_2 - U_1^* \widetilde{U}_2 \widetilde{\Sigma}_2 &= \Sigma_1 V_1^*(I - D_2)\widetilde{V}_2 + U_1^*(D_1^{-*} - I)\widetilde{U}_2\widetilde{\Sigma}_2, \\ \Sigma_1 U_1^* \widetilde{U}_2 - V_1^* \widetilde{V}_2 \widetilde{\Sigma}_2 &= \Sigma_1 U_1^*(I - D_1)\widetilde{U}_2 + V_1^*(D_2^{-*} - I)\widetilde{V}_2\widetilde{\Sigma}_2 \end{aligned}$$

which yield

$$V_1^* \widetilde{V}_2 - \Sigma_1^{-1} U_1^* \widetilde{U}_2 \widetilde{\Sigma}_2 = V_1^*(I - D_2)\widetilde{V}_2 + \Sigma_1^{-1} U_1^*(D_1^{-*} - I)\widetilde{U}_2\widetilde{\Sigma}_2, \quad (10.7)$$

$$U_1^* \widetilde{U}_2 - \Sigma_1^{-1} V_1^* \widetilde{V}_2 \widetilde{\Sigma}_2 = U_1^*(I - D_1)\widetilde{U}_2 + \Sigma_1^{-1} V_1^*(D_2^{-*} - I)\widetilde{V}_2\widetilde{\Sigma}_2. \quad (10.8)$$

Lemma 9.4 implies that $\left\|U_1^*\widetilde{U}_2\right\|_2 = \left\|V_1^*\widetilde{V}_2\right\|_2 = 1$, since $U_1^*\widetilde{U}_2$ is a $k\times(n-k+1)$ submatrix of $U^*\widetilde{U} \in \mathbb{U}_n$ and $V_1^*\widetilde{V}_2$ is a $k\times(n-k+1)$ submatrix of $V^*\widetilde{V} \in \mathbb{U}_n$ and $k + (n-k+1) = n+1 > n$. So it follows from (10.7) that

$$\begin{aligned} 1 - \frac{\widetilde{\sigma}_k}{\sigma_k} &= \left\|V_1^*\widetilde{V}_2\right\|_2 - \|\Sigma_1^{-1}\|_2 \left\|U_1^*\widetilde{U}_2\right\|_2 \|\widetilde{\Sigma}_2\|_2 \\ &\le \left\|V_1^*\widetilde{V}_2\right\|_2 - \left\|\Sigma_1^{-1}U_1^*\widetilde{U}_2\widetilde{\Sigma}_2\right\|_2 \\ &\le \left\|V_1^*\widetilde{V}_2 - \Sigma_1^{-1}U_1^*\widetilde{U}_2\widetilde{\Sigma}_2\right\|_2 \\ &= \left\|V_1^*(I - D_2)\widetilde{V}_2 + \Sigma_1^{-1}U_1^*(D_1^{-*} - I)\widetilde{U}_2\widetilde{\Sigma}_2\right\|_2 \\ &\le \|I - D_2\|_2 + \frac{\widetilde{\sigma}_k}{\sigma_k}\|D_1^{-*} - I\|_2 \\ &\le \sqrt[p]{1 + \frac{\widetilde{\sigma}_k^p}{\sigma_k^p}}\sqrt[q]{\|I - D_2\|_2^q + \|D_1^{-*} - I\|_2^q}. \end{aligned}$$

Therefore

$$\eta_p = \frac{1 - \widetilde{\sigma}_k/\sigma_k}{\sqrt[p]{1 + \widetilde{\sigma}_k^p/\sigma_k^p}} \le \sqrt[q]{\|I - D_2\|_2^q + \|D_1^{-*} - I\|_2^q}.$$

Similarly, it follows from (10.8) that

$$\eta_p = \frac{1 - \widetilde{\sigma}_k/\sigma_k}{\sqrt[p]{1 + \widetilde{\sigma}_k^p/\sigma_k^p}} \le \sqrt[q]{\|I - D_1\|_2^q + \|D_2^{-*} - I\|_2^q}.$$

The inequality (6.16) is a consequence of the last two inequalities. ∎

11 Proof of Theorems 5.1, 5.2, 5.3 and 5.4

Proof of Theorem 5.1: Since A is nonnegative, there is a matrix $B \in \mathbb{C}^{n\times n}$ such that $A = B^*B$. With this B, $\widetilde{A} = D^*AD = D^*B^*BD = \widetilde{B}^*\widetilde{B}$, where $\widetilde{B} = BD$. Let SVDs of B and $\widetilde{B}$ be as

$$B = U\Lambda^{1/2}V^* \quad \text{and} \quad \widetilde{B} = \widetilde{U}\widetilde{\Lambda}^{1/2}\widetilde{V}^*,$$

where

$$\Lambda^{1/2} = \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}) \quad \text{and} \quad \widetilde{\Lambda}^{1/2} = \mathrm{diag}\left(\sqrt{\widetilde{\lambda}_1}, \cdots, \sqrt{\widetilde{\lambda}_n}\right).$$

In what follows, we actually work with BB^* and $\widetilde{B}\widetilde{B}^*$, instead of $A = B^*B$ and $\widetilde{A} = \widetilde{B}^*\widetilde{B}$.

$$\begin{aligned} \widetilde{B}\widetilde{B}^* - BB^* &= \widetilde{B}D^*B^* - \widetilde{B}D^{-1}B^* \\ &= \widetilde{B}(D^* - D^{-1})B^*, \\ \widetilde{U}^*(\widetilde{B}\widetilde{B}^* - BB^*)U &= \widetilde{\Lambda}\widetilde{U}^*U - \widetilde{U}^*U\Lambda, \\ \widetilde{U}^*\widetilde{B}(D^* - D^{-1})B^*U &= \widetilde{\Lambda}^{1/2}\widetilde{V}^*(D^* - D^{-1})V\Lambda^{1/2}. \end{aligned}$$

Thus, we have the following perturbation equation.

$$\widetilde{\Lambda}\widetilde{U}^*U - \widetilde{U}^*U\Lambda = \widetilde{\Lambda}^{1/2}\widetilde{V}^*(D^* - D^{-1})V\Lambda^{1/2}. \tag{11.1}$$

Write $Q \stackrel{\text{def}}{=} \widetilde{U}^*U = (q_{ij})$. It follows from (11.1) that

$$\|\widetilde{V}^*(D^* - D^{-1})V\|_F^2 = \|D^* - D^{-1}\|_F^2 \geq \sum_{i,j=1}^{n} \frac{|\widetilde{\lambda}_i - \lambda_j|}{\sqrt{\widetilde{\lambda}_i\lambda_j}}|q_{ij}|^2.$$

Since $(|q_{ij}|^2)$ is a doubly stochastic matrix, applying Lemma 9.2 concludes the proof of the inequality (5.2). To show (5.1), let k be the index such that

$$\eta_p \stackrel{\text{def}}{=} \max_{1\leq i\leq n} \widetilde{\mathrm{RelDist}}(\lambda_i, \widetilde{\lambda}_i) = \widetilde{\mathrm{RelDist}}(\lambda_k, \widetilde{\lambda}_k).$$

If $\eta_p = 0$, no proof is necessary. Assume $\eta_p > 0$. Also assume, without lose of any generality, that

$$\lambda_k > \widetilde{\lambda}_k \geq 0.$$

Partition U, V, $\widetilde{U}$, $\widetilde{V}$ as follows

$$U = (U_1, U_2),\ V = (V_1, V_2),\ \widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2) \quad \text{and} \quad \widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2),$$

where U_1, $V_1 \in \mathbb{C}^{n\times k}$ and $\widetilde{U}_1$, $\widetilde{V}_1 \in \mathbb{C}^{n\times(k-1)}$, and write $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ and $\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2)$, where $\Lambda_1 \in \mathbb{R}^{k\times k}$ and $\widetilde{\Lambda}_1 \in \mathbb{R}^{(k-1)\times(k-1)}$. It follows from the equation (11.1) that

$$\widetilde{\Lambda}_2\widetilde{U}_2^*U_1 - \widetilde{U}_2^*U_1\Lambda_1 = \widetilde{\Lambda}_2^{1/2}\widetilde{V}_2^*(D^* - D^{-1})V_1\Lambda_1^{1/2}$$

which yields

$$\widetilde{\Lambda}_2\widetilde{U}_2^*U_1\Lambda_1^{-1} - \widetilde{U}_2^*U_1 = \widetilde{\Lambda}_2^{1/2}\widetilde{V}_2^*(D^* - D^{-1})V_1\Lambda_1^{-1/2}. \tag{11.2}$$

Lemma 9.4 implies that $\left\|\widetilde{U}_2^*U_1\right\|_2 = 1$ since $\widetilde{U}_2^*U_1$ is a $(n-k+1)\times k$ submatrix of $\widetilde{U}^*U$ and $k+(n-k+1) = n+1 > n$. So it follows from (11.2) that

$$\begin{aligned}
1 - \frac{\widetilde{\lambda}_k}{\lambda_k} &= \left\|\widetilde{U}_2^*U_1\right\|_2 - \|\widetilde{\Lambda}_2\|_2\left\|\widetilde{U}_2^*U_1\right\|_2\|\Lambda_1^{-1}\|_2 \\
&\leq \left\|\widetilde{U}_2^*U_1\right\|_2 - \left\|\widetilde{\Lambda}_2\widetilde{U}_2^*U_1\Lambda_1^{-1}\right\|_2 \\
&\leq \left\|\widetilde{U}_2^*U_1 - \widetilde{\Lambda}_2\widetilde{U}_2^*U_1\Lambda_1^{-1}\right\|_2 \\
&= \left\|\widetilde{\Lambda}_2^{1/2}\widetilde{V}_2^*(D^* - D^{-1})V_1\Lambda_1^{-1/2}\right\|_2 \\
&\leq \|\widetilde{\Lambda}_2^{1/2}\|_2\left\|\widetilde{V}_2^*(D^* - D^{-1})V_1\right\|_2\|\Lambda_1^{-1/2}\|_2 \\
&= \sqrt{\frac{\widetilde{\lambda}_k}{\lambda_k}}\left\|\widetilde{V}_2^*(D^* - D^{-1})V_1\right\|_2 \\
&\leq \sqrt{\frac{\widetilde{\lambda}_k}{\lambda_k}}\|D^* - D^{-1}\|_2,
\end{aligned}$$

an immediate consequence of which is the inequality (5.1). ∎

Proof of Theorem 5.2: Set $\widehat{B} = BD_2$ and denote

$$\sigma(\widehat{B}) = \{\widehat{\sigma}_1 \geq \widehat{\sigma}_2 \geq \cdots \geq \widehat{\sigma}_n\}.$$

Applying Theorem 5.1 to B^*B and $\widehat{B}^*\widehat{B} = D_2^*B^*BD_2$ leads to

$$\begin{aligned}
\max_{1\leq i\leq n} \widetilde{\mathrm{RelDist}}(\sigma_i^2, \widehat{\sigma}_i^2) &\leq \|D_2^* - D_2^{-1}\|_2, \\
\sqrt{\sum_{i=1}^{n}\left[\widetilde{\mathrm{RelDist}}(\sigma_i^2, \widehat{\sigma}_i^2)\right]^2} &\leq \|D_2^* - D_2^{-1}\|_F.
\end{aligned}$$

Now applying Proposition 2.11, we obtain

$$\begin{aligned}
\max_{1\leq i\leq n} \widetilde{\mathrm{RelDist}}(\sigma_i, \widehat{\sigma}_i) &\leq \frac{1}{2}\|D_2^* - D_2^{-1}\|_2, \\
\sqrt{\sum_{i=1}^{n}\left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widehat{\sigma}_i)\right]^2} &\leq \frac{1}{2}\|D_2^* - D_2^{-1}\|_F.
\end{aligned}$$

Similarly for BD_2 and $D_1^* BD_2$, we have

$$\begin{aligned} \max_{1\le i\le n} \widetilde{\mathrm{RelDist}}(\widehat{\sigma}_i, \widetilde{\sigma}_i) &\le \frac{1}{2}\|D_1^* - D_1^{-1}\|_2, \\ \sqrt{\sum_{i=1}^n \left[\widetilde{\mathrm{RelDist}}(\widehat{\sigma}_i, \widetilde{\sigma}_i)\right]^2} &\le \frac{1}{2}\|D_1^* - D_1^{-1}\|_F. \end{aligned}$$

Since $\widetilde{\mathrm{RelDist}}$ is a generalized metric on $\mathbb{R}_{\ge 0}$, we get

$$\begin{aligned} \widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i) &\le \widetilde{\mathrm{RelDist}}(\sigma_i, \widehat{\sigma}_i) + \widetilde{\mathrm{RelDist}}(\widehat{\sigma}_i, \widetilde{\sigma}_i) \\ &\le \frac{1}{2}\left(\|D_1^* - D_1^{-1}\|_2 + \|D_2^* - D_2^{-1}\|_2\right), \\ \sqrt{\sum_{i=1}^n \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widetilde{\sigma}_i)\right]^2} &\le \sqrt{\sum_{i=1}^n \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widehat{\sigma}_i) + \widetilde{\mathrm{RelDist}}(\widehat{\sigma}_i, \widetilde{\sigma}_i)\right]^2} \\ &\le \sqrt{\sum_{i=1}^n \left[\widetilde{\mathrm{RelDist}}(\sigma_i, \widehat{\sigma}_i)\right]^2} + \sqrt{\sum_{i=1}^n \left[\widetilde{\mathrm{RelDist}}(\widehat{\sigma}_i, \widetilde{\sigma}_i)\right]^2} \\ &\le \frac{1}{2}\left(\|D_1^* - D_1^{-1}\|_F + \|D_2^* - D_2^{-1}\|_F\right), \end{aligned}$$

as expected. ∎

Proof of Theorem 5.3*:* Write

$$\widetilde{G} = (B + \Delta B)D = (I + (\Delta B)B^{-1})BD = \widehat{D}G,$$

where $\widehat{D} = I + (\Delta B)B^{-1}$. Now applying Theorem 5.2 above to G and $\widetilde{G} = \widehat{D}G$ yields the first inequalities in both (5.9) and in (5.10). To get the second inequalities, we notice

$$(I+E)^* - (I+E)^{-1} = I + E^* - \sum_{i=0}^{\infty}(-1)^i E^i = E^* + E + E\sum_{i=2}^{\infty}(-1)^i E^{i-1},$$

where $E = (\Delta B)B^{-1}$, and therefore for any unitarily invariant norm $|||\cdot|||$

$$\begin{aligned} \left|\left|\left|(I+E)^* - (I+E)^{-1}\right|\right|\right| &\le |||E + E^*||| + |||E|||\sum_{i=1}^{\infty}\|E\|_2^i \\ &= \left(\frac{|||E+E^*|||}{|||E|||} + \frac{\|E\|_2}{1 - \|E\|_2}\right)|||E|||. \end{aligned}$$

The rest is trivial. ∎

Proof of Theorem 5.4*:* Rewrite H and $\widetilde{H}$ as

$$
\begin{aligned}
H &= D^*AD = (A^{1/2}D)^*\, A^{1/2}D \stackrel{\text{def}}{=} B^*B, \\
\widetilde{H} &= D^*A^{1/2}\big(I + A^{-1/2}(\Delta A)A^{-1/2}\big)A^{1/2}D \\
&= \Big(\big(I + A^{-1/2}(\Delta A)A^{-1/2}\big)^{1/2}A^{1/2}D\Big)^*\ \big(I + A^{-1/2}(\Delta A)A^{-1/2}\big)^{1/2}A^{1/2}D \\
&\stackrel{\text{def}}{=} \widetilde{B}^*\widetilde{B},
\end{aligned}
$$

where

$$
\begin{aligned}
B &\stackrel{\text{def}}{=} A^{1/2}D, \\
\widetilde{B} &\stackrel{\text{def}}{=} \big(I + A^{-1/2}(\Delta A)A^{-1/2}\big)^{1/2}A^{1/2}D.
\end{aligned}
$$

Set $\widehat{D} = \big(I + A^{-1/2}(\Delta A)A^{-1/2}\big)^{1/2}$. Thus $\widetilde{B} = \widehat{D}B$. Notice that $\lambda(H) = \lambda(B^*B) = \lambda(BB^*)$ and $\lambda(\widetilde{H}) = \lambda(\widetilde{B}^*\widetilde{B}) = \lambda(\widetilde{B}\widetilde{B}^*)$ and $\widetilde{B}\widetilde{B}^* = \widehat{D}BB^*\widehat{D}^*$. So applying Theorem 5.1 to BB^* and $\widetilde{B}\widetilde{B}^*$ yields the first "$\leq$" in both (5.13) and (5.14). ∎

12 Proof of Theorem 6.6

There is nothing to prove if $\widetilde{\lambda} \in \lambda(A)$. Assume that $\widetilde{\lambda} \notin \lambda(A)$. Here we will prove the case when $\widetilde{A} = DA$ only, since the proof for the case when $\widetilde{A} = AD$ is very similar. Consider $\widetilde{A} - \widetilde{\lambda}I$.

$$\begin{aligned} \widetilde{A} - \widetilde{\lambda}I &= A - \widetilde{\lambda}I + \widetilde{A} - A \\ &= X(\Lambda - \widetilde{\lambda}I)X^{-1} + (D - I)X\Lambda X^{-1} \\ &= X\left[I + X^{-1}(D - I)X\Lambda(\Lambda - \widetilde{\lambda}I)^{-1}\right](\Lambda - \widetilde{\lambda}I)X^{-1}. \end{aligned}$$

Since $\widetilde{A} - \widetilde{\lambda}I$ is singular, we have for any $1 \le p \le \infty$

$$\|X^{-1}(D - I)X\Lambda(\Lambda - \widetilde{\lambda}I)^{-1}\|_p \ge 1$$

which gives

$$1 \le \|X^{-1}(D - I)X\|_p \|\Lambda(\Lambda - \widetilde{\lambda}I)^{-1}\|_p = \|X^{-1}(D - I)X\|_p \max_{\lambda \in \lambda(A)} \frac{|\lambda|}{|\widetilde{\lambda} - \lambda|}$$

as was to be shown. ∎

A Is $\mathrm{RelDist}_p$ a Metric?

In this appendix, we will prove (2.22) under certain conditions. As a result, we will see

1. $\mathrm{RelDist}_p$ is a metric on $\mathbb{R}_{\geq 0}$;
2. $\mathrm{RelDist}_1$, $\mathrm{RelDist}_2$ and $\mathrm{RelDist}_\infty$ are metrics on $\mathbb{R}$.

We strongly conjecture that $\mathrm{RelDist}_p$ is a metric on $\mathbb{C}$. Unfortunately, we are unable to prove it at this point.

Lemma A.1 *The following statements are equivalent:*

1. $\mathrm{RelDist}_p(\alpha, \gamma) \leq \mathrm{RelDist}_p(\alpha, \beta) + \mathrm{RelDist}_p(\beta, \gamma)$;
2. $\mathrm{RelDist}_p(\xi\alpha, \xi\gamma) \leq \mathrm{RelDist}_p(\xi\alpha, \xi\beta) + \mathrm{RelDist}_p(\xi\beta, \xi\gamma)$ *for some* $0 \neq \xi \in \mathbb{C}$;
3. $\mathrm{RelDist}_p(\xi\alpha, \xi\gamma) \leq \mathrm{RelDist}_p(\xi\alpha, \xi\beta) + \mathrm{RelDist}_p(\xi\beta, \xi\gamma)$ *for all* $0 \neq \xi \in \mathbb{C}$.

The proof of this lemma is trivial, just by Property 3 of Proposition 2.1. With Lemma A.1 in mind and that swapping α and γ does not lose any generality, we may assume from now on

$$\alpha \leq |\alpha| \leq \gamma. \tag{A.1}$$

The inequality (2.22) is trivial when one of the α, β, γ is zero or $\beta = \alpha$ or $\beta = \gamma$. So from now on, we may assume

$$\alpha,\ \beta,\ \gamma \neq 0 \quad \text{and} \quad \alpha \neq \beta \neq \gamma. \tag{A.2}$$

Now there are three possible positions for β:

$$\beta < \alpha \quad \text{or} \quad \alpha < \beta < \gamma \quad \text{or} \quad \gamma < \beta.$$

When $\alpha < 0$, we split the case $\beta < \alpha$ into two subcases:

$$\beta < -\gamma \quad \text{or} \quad -\gamma \leq \beta < \alpha.$$

Also in the case $\alpha < 0$, without loss of generality, we may assume $\alpha = -1$ by Lemma A.1. We summarize the above cases we have to handle separately as follows.

1. $\alpha < \beta < \gamma$;
2. $\alpha\gamma > 0$, i.e., α and γ are of the same sign;
3. $\alpha = -1 < 1 \leq \gamma < \beta$;

4. $-\gamma \leq \beta < \alpha = -1 < 1 \leq \gamma$; and

5. $\beta \leq -\gamma \leq \alpha = -1 < 1 \leq \gamma$.

Lemma A.2 *(2.22) holds for $\alpha \leq \beta \leq \gamma$, and the equality sign holds if and only if $\beta = \alpha$ or $\beta = \gamma$.*

This lemma actually implies that (2.22) holds if β lies between α and γ for all $\alpha, \gamma \in \mathbb{R}$, not just these α and γ satisfying (A.1).

Proof: Assume $\alpha \neq \beta \neq \gamma$. Because of (A.1), we have $\gamma - \alpha = \gamma - \beta + \beta - \alpha$ and thus for $1 \leq p < \infty$

$$\begin{aligned}
\mathrm{RelDist}_p(\alpha,\gamma) &= \frac{\gamma-\alpha}{\sqrt[p]{\gamma^p + |\alpha|^p}} = \frac{\gamma-\beta}{\sqrt[p]{\gamma^p + |\alpha|^p}} + \frac{\beta-\alpha}{\sqrt[p]{\gamma^p + |\alpha|^p}} \\
&= \frac{\gamma-\beta}{\sqrt[p]{\gamma^p + |\beta|^p}} + \frac{\beta-\alpha}{\sqrt[p]{|\beta|^p + |\alpha|^p}} \\
&\quad +(\gamma-\beta)\left(\frac{1}{\sqrt[p]{\gamma^p + |\alpha|^p}} - \frac{1}{\sqrt[p]{\gamma^p + |\beta|^p}}\right) \\
&\quad +(\beta-\alpha)\left(\frac{1}{\sqrt[p]{\gamma^p + |\alpha|^p}} - \frac{1}{\sqrt[p]{|\alpha|^p + |\beta|^p}}\right) \\
&= \mathrm{RelDist}_p(\alpha,\beta) + \mathrm{RelDist}_p(\beta,\gamma) \\
&\quad + \frac{(\gamma-\beta)(|\beta|^p - |\alpha|^p)}{\sqrt[p]{\gamma^p + |\alpha|^p}\,\sqrt[p]{\gamma^p + |\beta|^p}} \cdot \frac{\sqrt[p]{\gamma^p + |\beta|^p} - \sqrt[p]{\gamma^p + |\alpha|^p}}{|\beta|^p - |\alpha|^p} \\
&\quad + \frac{(\beta-\alpha)(|\beta|^p - \gamma^p)}{\sqrt[p]{\gamma^p + |\alpha|^p}\,\sqrt[p]{|\alpha|^p + |\beta|^p}} \cdot \frac{\sqrt[p]{|\alpha|^p + |\beta|^p} - \sqrt[p]{\gamma^p + |\alpha|^p}}{|\beta|^p - \gamma^p}.
\end{aligned}$$

Now if $\alpha < \beta \leq |\alpha| \leq \gamma$, then $|\beta|^p - |\alpha|^p \leq 0$ and $|\beta|^p - \gamma^p < 0$, and thus

$$\begin{aligned}
&\frac{(\gamma-\beta)(|\beta|^p - |\alpha|^p)}{\sqrt[p]{\gamma^p + |\alpha|^p}\,\sqrt[p]{\gamma^p + |\beta|^p}} \cdot \frac{\sqrt[p]{\gamma^p + |\beta|^p} - \sqrt[p]{\gamma^p + |\alpha|^p}}{|\beta|^p - |\alpha|^p} \\
&\quad + \frac{(\beta-\alpha)(|\beta|^p - \gamma^p)}{\sqrt[p]{\gamma^p + |\alpha|^p}\,\sqrt[p]{|\alpha|^p + |\beta|^p}} \cdot \frac{\sqrt[p]{|\alpha|^p + |\beta|^p} - \sqrt[p]{\gamma^p + |\alpha|^p}}{|\beta|^p - \gamma^p} < 0.
\end{aligned}$$

Hence $\mathrm{RelDist}_p(\alpha,\gamma) < \mathrm{RelDist}_p(\alpha,\beta) + \mathrm{RelDist}_p(\beta,\gamma)$. Consider now $|\alpha| < \beta < \gamma$. Then

$$\begin{aligned}
&\frac{(\gamma-\beta)(|\beta|^p - |\alpha|^p)}{\sqrt[p]{\gamma^p + |\alpha|^p}\,\sqrt[p]{\gamma^p + |\beta|^p}} \cdot \frac{\sqrt[p]{\gamma^p + |\beta|^p} - \sqrt[p]{\gamma^p + |\alpha|^p}}{|\beta|^p - |\alpha|^p} \\
&\quad + \frac{(\beta-\alpha)(|\beta|^p - \gamma^p)}{\sqrt[p]{\gamma^p + |\alpha|^p}\,\sqrt[p]{|\alpha|^p + |\beta|^p}} \cdot \frac{\sqrt[p]{|\alpha|^p + |\beta|^p} - \sqrt[p]{\gamma^p + |\alpha|^p}}{|\beta|^p - \gamma^p}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{(\gamma-\beta)(\beta-|\alpha|)}{\sqrt[p]{\gamma^p+|\alpha|^p}}\left(\frac{1}{\sqrt[p]{\gamma^p+\beta^p}}\cdot\frac{\beta^p-|\alpha|^p}{\beta-|\alpha|}\cdot\frac{\sqrt[p]{\gamma^p+\beta^p}-\sqrt[p]{\gamma^p+|\alpha|^p}}{\beta^p-|\alpha|^p}\right.\\
&\qquad\left.-\frac{1}{\sqrt[p]{|\alpha|^p+\beta^p}}\cdot\frac{\gamma^p-\beta^p}{\gamma-\beta}\cdot\frac{\sqrt[p]{|\alpha|^p+\beta^p}-\sqrt[p]{\gamma^p+|\alpha|^p}}{\beta^p-\gamma^p}\right)\\
&< 0.
\end{aligned}
$$

The last "<" is true because $\sqrt[p]{\gamma^p+\beta^p} > \sqrt[p]{|\alpha|^p+\beta^p}$ and

$$
\begin{aligned}
0 &< \frac{\beta^p-|\alpha|^p}{\beta-|\alpha|}\le\frac{\gamma^p-\beta^p}{\gamma-\beta},\\
0 &< \frac{\sqrt[p]{\gamma^p+\beta^p}-\sqrt[p]{\gamma^p+|\alpha|^p}}{\beta^p-|\alpha|^p}\le\frac{\sqrt[p]{|\alpha|^p+\beta^p}-\sqrt[p]{\gamma^p+|\alpha|^p}}{\beta^p-\gamma^p}.
\end{aligned}
$$

So we also have $\mathrm{RelDist}_p(\alpha,\gamma) < \mathrm{RelDist}_p(\alpha,\beta)+\mathrm{RelDist}_p(\beta,\gamma)$ for $|\alpha|<\beta<\gamma$. The proof for the case $p<\infty$ is completed.

When $p=\infty$,

$$
\begin{aligned}
\mathrm{RelDist}_\infty(\alpha,\gamma) &= \frac{\gamma-\alpha}{\gamma}=\frac{\gamma-\beta}{\gamma}+\frac{\beta-\alpha}{\gamma}\\
&= \frac{\gamma-\beta}{\gamma}+\frac{\beta-\alpha}{\max\{|\alpha|,|\beta|\}}\\
&\quad +(\beta-\alpha)\left(\frac{1}{\gamma}-\frac{1}{\max\{|\alpha|,|\beta|\}}\right)\\
&< \mathrm{RelDist}_\infty(\alpha,\beta)+\mathrm{RelDist}_\infty(\beta,\gamma),
\end{aligned}
$$

as was to be shown. ∎

Lemma A.3 *(2.22) holds for* $\alpha\gamma\ge 0$*.*

Proof: Lemma A.2 shows that (2.22) is true if $\alpha\le\beta\le\gamma$. If either $\beta<\alpha$ or $\gamma<\beta$, (2.22) follows from Property 8 of Proposition 2.1. ∎

As an immediate consequence of Lemma A.3, we have

Proposition A.1 $\mathrm{RelDist}_p$ *is a metric on* $\mathbb{R}_{\ge 0}$*.*

Lemma A.4 *(2.22) holds for* $-\gamma\le\beta\le\alpha<0<|\alpha|\le\gamma$*, and the equality sign holds if and only if* $\beta=\alpha$*.*

Proof: Assume $\beta\ne\alpha$. Define

$$
f(\xi)\overset{\mathrm{def}}{=}\frac{\gamma+\xi}{\sqrt[p]{\gamma^p+\xi^p}}\quad\text{for } |\alpha|\le\xi\le\gamma.
$$

Clearly if $p = \infty$, $f(\xi) = \frac{\gamma+\xi}{\gamma}$ increases in $|\alpha| \le \xi \le \gamma$; if $0 \le p < \infty$, we have

$$f'(\xi) = \frac{\gamma(\gamma^{p-1} - \xi^{p-1})}{(\gamma^p + \xi^p)^{1+1/p}} > 0, \quad \text{for } |\alpha| \le \xi < \gamma.$$

So $f(\xi)$ is an increasing function for all p. Hence

$$\mathrm{RelDist}_p(\alpha,\gamma) = f(-\alpha) < f(-\beta) = \mathrm{RelDist}_p(\beta,\gamma) < \mathrm{RelDist}_p(\alpha,\beta) + \mathrm{RelDist}_p(\beta,\gamma),$$

as was to be proved. ∎

Proposition A.2 RelDist$_1$, RelDist$_2$ *and* RelDist$_\infty$ *are metrics on* $\mathbb{R}$*.*

Proof: We have to prove (2.22) with $p = 1, 2$ and ∞ for all 5 cases listed at the beginning of this appendix. But *Case 1* has been covered by Lemma A.2, *Case 2* by Lemma A.3, and *Case 4* by Lemma A.4. *Cases 3* and *5* are to be dealt with by Lemmas A.5 and A.6 below.

Lemma A.5 *(2.22) with* $p = 1, 2$ *or* ∞ *holds for* $\alpha = -1 < 1 \le \gamma \le \beta$*. When* $p = 1$, 2*, the equality sign holds if and only if* $\beta = \gamma$*; when* $p = \infty$*, the equality sign holds if and only if either* $\beta = \gamma$ *or* $\gamma = 1$*.*

Proof: Assume $\beta \ne \gamma$. First consider the case $p = 2$. Define

$$f(\xi) \stackrel{\text{def}}{=} \frac{\xi+1}{\sqrt{\xi^2+1}} + \frac{\xi-\gamma}{\sqrt{\xi^2+\gamma^2}}.$$

We are going to show that $f'(\xi) > 0$ for $\xi > \gamma$ and thus

$$\mathrm{RelDist}_2(-1,\beta) + \mathrm{RelDist}_2(\beta,\gamma) = f(\beta) > f(\gamma) = \frac{\gamma+1}{\sqrt{\gamma^2+1}} = \mathrm{RelDist}_2(-1,\gamma)$$

which concludes the proof for the present case. Since

$$f'(\xi) = -\frac{\xi-1}{(\xi^2+1)^{3/2}} + \frac{\gamma(\xi+\gamma)}{(\xi^2+\gamma^2)^{3/2}}.$$

So to show $f'(\xi) > 0$, it suffices for us to show for $\xi > \gamma \ge 1$

$$\gamma(\xi+\gamma)(\xi^2+1)^{3/2} > (\xi-1)(\xi^2+\gamma^2)^{3/2},$$

or equivalently, to show for $\xi > \gamma \ge 1$

$$(\xi-1)^2(\xi^2+\gamma^2)^3 - \gamma^2(\xi+\gamma)^2(\xi^2+1)^3 < 0.$$

But tedious algebraic manipulations yield the following

$$\begin{aligned}
&(\xi-1)^2(\xi^2+\gamma^2)^3-\gamma^2(\xi+\gamma)^2(\xi^2+1)^3\\
&= -\gamma^4+\gamma^6-2\gamma^3(1+\gamma^3)\xi+\gamma^2(\gamma^4-1)\xi^2-6\gamma^3(1+\gamma)\xi^3\\
&\quad -6\gamma^2(1+\gamma)\xi^5+(1-\gamma^4)\xi^6-2(1+\gamma^3)\xi^7+(1-\gamma^2)\xi^8\\
&= -\gamma^4+\gamma^3(\gamma^3-6\xi^3)-2\gamma^3(1+\gamma^3)\xi-\gamma^2\xi^2+\gamma^3\xi^2(\gamma^3-6\xi^3)\\
&\quad -6\gamma^4\xi^3-6\gamma^2\xi^5+(1-\gamma^4)\xi^6-2(1+\gamma^3)\xi^7+(1-\gamma^2)\xi^8\\
&< 0,
\end{aligned}$$

as required. This completes the proof for $p=2$.

We have to show (2.22) for $p=1$ or ∞. For the moment, let's see what is the implication of (2.22) for any $1\le p\le\infty$ for this particular case. Notice $\gamma+1=\beta+1-(\beta-\gamma)$ and

$$\begin{aligned}
\mathrm{RelDist}_p(-1,\gamma) &= \frac{\gamma+1}{\sqrt[p]{\gamma^p+1}} = \frac{\beta+1}{\sqrt[p]{\gamma^p+1}} - \frac{\beta-\gamma}{\sqrt[p]{\gamma^p+1}}\\
&= \frac{\beta+1}{\sqrt[p]{\beta^p+1}} + \frac{\beta-\gamma}{\sqrt[p]{\beta^p+\gamma^p}}\\
&\quad +(\beta+1)\left(\frac{1}{\sqrt[p]{\gamma^p+1}}-\frac{1}{\sqrt[p]{\beta^p+1}}\right)-(\beta-\gamma)\left(\frac{1}{\sqrt[p]{\gamma^p+1}}+\frac{1}{\sqrt[p]{\beta^p+\gamma^p}}\right)\\
&= \mathrm{RelDist}_p(-1,\beta)+\mathrm{RelDist}_p(\beta,\gamma)\\
&\quad +(\beta+1)\frac{\sqrt[p]{\beta^p+1}-\sqrt[p]{\gamma^p+1}}{\sqrt[p]{\gamma^p+1}\sqrt[p]{\beta^p+1}}-(\beta-\gamma)\frac{\sqrt[p]{\beta^p+\gamma^p}+\sqrt[p]{\gamma^p+1}}{\sqrt[p]{\gamma^p+1}\sqrt[p]{\beta^p+\gamma^p}}.
\end{aligned}$$

So (2.22) holds if and only if

$$\begin{aligned}
&(\beta+1)\left(\sqrt[p]{\beta^p+1}-\sqrt[p]{\gamma^p+1}\right)\sqrt[p]{\beta^p+\gamma^p}\\
&\le (\beta-\gamma)\left(\sqrt[p]{\beta^p+\gamma^p}+\sqrt[p]{\gamma^p+1}\right)\sqrt[p]{\beta^p+1},
\end{aligned}$$

or equivalently

$$\sqrt[p]{\beta^p+\gamma^p}\left((\gamma+1)\sqrt[p]{\beta^p+1}-(\beta+1)\sqrt[p]{\gamma^p+1}\right)\le(\beta-\gamma)\sqrt[p]{\beta^p+\gamma^p}\sqrt[p]{\beta^p+1}$$

which is true if and only if

$$\sqrt[p]{\beta^p+\gamma^p}\left(\frac{\gamma+1}{\sqrt[p]{\gamma^p+1}}-\frac{\beta+1}{\sqrt[p]{\beta^p+1}}\right)\le\beta-\gamma. \tag{A.3}$$

Our proof will be completed if we can prove (A.3) for $p=1$ or ∞. When $p=1$, the left-hand side of (A.3) is zero and its right-hand side is $\beta-\gamma\ge 0$. When $p=\infty$,

$$\text{the left-hand side of (A.3)} = \beta\left(\frac{\gamma+1}{\gamma}-\frac{\beta+1}{\beta}\right)=\frac{\beta-\gamma}{\gamma}\le\beta-\gamma.$$

Hence (A.3) holds for both $p=1$ and ∞. ∎

Lemma A.6 *(2.22) with $p = 1, 2$ or ∞ holds for $\beta < -\gamma \leq \alpha = -1 < 1 \leq \gamma$, and is strict, unless $p = \infty$ and $\gamma = 1$.*

Proof: We want to prove for $p = 1$, 2 and ∞

$$\mathrm{RelDist}_p(-1,\gamma) < \mathrm{RelDist}_p(-1,\beta) + \mathrm{RelDist}_p(\beta,\gamma),$$

which, by Lemma A.1, is equivalent to

$$\mathrm{RelDist}_p(1,-\gamma) < \mathrm{RelDist}_p(1,-\beta) + \mathrm{RelDist}_p(-\beta,-\gamma).$$

Set $\xi = -\beta$. Then $\xi > \gamma > 1$. For the moment, let's see what is the implication of (2.22) for any $1 \leq p \leq \infty$ for this particular case. Notice that $\gamma + 1 = \xi + \gamma - (\xi - 1)$, and thus

$$\begin{aligned}
\mathrm{RelDist}_p(1,-\gamma) &= \frac{\gamma+1}{\sqrt[p]{\gamma^p+1}} = \frac{\xi+\gamma}{\sqrt[p]{\gamma^p+1}} - \frac{\xi-1}{\sqrt[p]{\gamma^p+1}} \\
&= \frac{\xi+\gamma}{\sqrt[p]{\xi^p+\gamma^p}} + \frac{\xi-1}{\sqrt[p]{\xi^p+1}} \\
&\quad +(\xi+\gamma)\left(\frac{1}{\sqrt[p]{\gamma^p+1}} - \frac{1}{\sqrt[p]{\xi^p+\gamma^p}}\right) - (\xi-1)\left(\frac{1}{\sqrt[p]{\gamma^p+1}} + \frac{1}{\sqrt[p]{\xi^p+1}}\right) \\
&\quad +(\xi+\gamma)\frac{\sqrt[p]{\xi^p+\gamma^p} - \sqrt[p]{\gamma^p+1}}{\sqrt[p]{\gamma^p+1}\sqrt[p]{\xi^p+\gamma^p}} - (\xi-1)\frac{\sqrt[p]{\xi^p+1} + \sqrt[p]{\gamma^p+1}}{\sqrt[p]{\gamma^p+1}\sqrt[p]{\xi^p+1}}.
\end{aligned}$$

So (2.22) holds if and only if

$$(\xi+\gamma)\frac{\sqrt[p]{\xi^p+\gamma^p} - \sqrt[p]{\gamma^p+1}}{\sqrt[p]{\gamma^p+1}\sqrt[p]{\xi^p+\gamma^p}} - (\xi-1)\frac{\sqrt[p]{\xi^p+1} + \sqrt[p]{\gamma^p+1}}{\sqrt[p]{\gamma^p+1}\sqrt[p]{\xi^p+1}} \leq 0, \tag{A.4}$$

or equivalently

$$(\gamma+\xi)\sqrt[p]{\xi^p+1}\left(\sqrt[p]{\gamma^p+\xi^p} - \sqrt[p]{\gamma^p+1}\right) \leq (\xi-1)\sqrt[p]{\gamma^p+\xi^p}\left(\sqrt[p]{\xi^p+1} + \sqrt[p]{\gamma^p+1}\right),$$

or equivalently

$$\sqrt[p]{\gamma^p+\xi^p}\left((\gamma+1)\sqrt[p]{\xi^p+1} - (\xi-1)\sqrt[p]{\gamma^p+1}\right) \leq (\gamma+\xi)\sqrt[p]{\gamma^p+1}\sqrt[p]{\xi^p+1}$$

which holds if and only if

$$\sqrt[p]{\gamma^p+\xi^p}\left(\frac{\gamma+1}{\sqrt[p]{\gamma^p+1}} - \frac{\xi-1}{\sqrt[p]{\xi^p+1}}\right) \leq \gamma+\xi. \tag{A.5}$$

We have to show (A.5) (or (A.4)) for $p = 1$, 2 and ∞. When $p = 2$, We will prove (A.4) by showing for $\xi > \gamma$

$$\frac{(\xi+\gamma)(\xi^2-1)}{\sqrt{\gamma^2+1}\sqrt{\xi^2+\gamma^2}\left(\sqrt{\gamma^2+1}+\sqrt{\xi^2+\gamma^2}\right)} - (\xi-1)\left(\frac{1}{\sqrt{\gamma^2+1}} + \frac{1}{\sqrt{\xi^2+1}}\right) < 0 \tag{A.6}$$

and thus our proof is completed. To show our claim, first, we notice that the inequality (A.6) is equivalent to

$$\frac{(\xi+\gamma)(\xi+1)}{\sqrt{\gamma^2+1}\sqrt{\xi^2+\gamma^2}\left(\sqrt{\gamma^2+1}+\sqrt{\xi^2+\gamma^2}\right)} < \frac{1}{\sqrt{\gamma^2+1}} + \frac{1}{\sqrt{\xi^2+1}},$$

or equivalently

$$\frac{(\xi+\gamma)(\xi+1)\sqrt{\xi^2+1}}{\sqrt{\xi^2+\gamma^2}} \le \left(\sqrt{\gamma^2+1}+\sqrt{\xi^2+\gamma^2}\right)\left(\sqrt{\xi^2+1}+\sqrt{\gamma^2+1}\right). \tag{A.7}$$

Notice that

$$\begin{aligned}
\text{The left-hand side of (A.7)} &\le (\xi+\gamma)(\xi+1), \\
\text{The right-hand side of (A.7)} &\ge \left(\sqrt{\xi^2+1}+\sqrt{\gamma^2+1}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sqrt{\xi^2+1}+\sqrt{\gamma^2+1}\right)^2 - (\xi+\gamma)(\xi+1) \\
&= \xi^2+1+\gamma^2+1+2\sqrt{\xi^2+1}\sqrt{\gamma^2+1}-\xi^2-(\gamma+1)\xi-\gamma \\
&> 2\sqrt{\xi^2+1}\sqrt{\gamma^2+1}-(\gamma+1)\xi \\
&> 0,
\end{aligned}$$

because

$$\begin{aligned}
(\gamma+1)^2\xi^2 &= \xi^2\gamma^2+2\xi^2\gamma+\xi^2 \\
&\le 3\xi^2\gamma^2+\xi^2, \\
4(\xi^2+1)(\gamma^2+1) &= 4\xi^2\gamma^2+4\xi^2+4\gamma^2+4.
\end{aligned}$$

Next, we are going to show (A.5) for $p=1$ and ∞. When $p=1$,

$$\text{the left-hand side of (A.5)} = (\gamma+\xi)\frac{2}{\xi+1} < \gamma+\xi;$$

When $p=\infty$,

$$\text{the left-hand side of (A.5)} = \frac{\xi+\gamma}{\gamma} \le \xi+\gamma,$$

and the equality sign holds if and only if $\gamma=1$.
By now the proof of that RelDist_1, RelDist_2 and RelDist_∞ are metrics on $\mathbb{R}$ is completed. ∎

We briefly summarize what we have proved in this appendix.

1. When $p = 1$, 2 or ∞, (2.22) is true for all α, β, $\gamma \in \mathbb{R}$, and thus $\mathrm{RelDist}_1$, $\mathrm{RelDist}_2$ and $\mathrm{RelDist}_\infty$ are metrics on $\mathbb{R}$;

2. (2.22) is true for all α, β, $\gamma \geq 0$ and for all $1 \leq p \leq \infty$, and thus $\mathrm{RelDist}_p$ for any $1 \leq p \leq \infty$ is a metric on $\mathbb{R}_{\geq 0}$;

3. (2.22) for $1 \leq p \leq \infty$ survives to *Case 1, Case 2* and *Case 4*. But we do not know whether it survives to *Case 3* and/or *Case 5*. We believe it would. Showing (2.22) survives to *Case 3* is equivalent to showing (A.3) for $1 \leq \gamma < \beta$; and showing (2.22) survives to *Case 5* is equivalent to showing (A.5) for $1 \leq \gamma \leq \xi$.

B Is $\widetilde{\text{RelDist}}$ a Generalized Metric?

In this appendix, we will prove (2.23) under certain conditions. As a result, we will see

1. $\widetilde{\text{RelDist}}$ is a generalized metric on $\mathbb{R}_{\geq 0}$, and a metric on $\mathbb{R}_+$;
2. $\widetilde{\text{RelDist}}$ is not a generalized metric on $\mathbb{R}$ (nor on $\mathbb{C}$, of course).

Similarly to Lemma A.1, we also have

Lemma B.1 *The following statements are equivalent:*

1. $\widetilde{\text{RelDist}}(\alpha,\gamma) \leq \widetilde{\text{RelDist}}(\alpha,\beta) + \widetilde{\text{RelDist}}(\beta,\gamma)$*;*
2. $\widetilde{\text{RelDist}}(\xi\alpha,\xi\gamma) \leq \widetilde{\text{RelDist}}(\xi\alpha,\xi\beta) + \widetilde{\text{RelDist}}(\xi\beta,\xi\gamma)$ *for some* $0 \neq \xi \in \mathbb{C}$*;*
3. $\widetilde{\text{RelDist}}(\xi\alpha,\xi\gamma) \leq \widetilde{\text{RelDist}}(\xi\alpha,\xi\beta) + \widetilde{\text{RelDist}}(\xi\beta,\xi\gamma)$ *for all* $0 \neq \xi \in \mathbb{C}$*.*

This lemma follows from Property 3 of Proposition 2.8. Again, now with Lemma B.1 in mind and that swapping α and γ does not lose any generality, we may assume (A.1) holds, and also only cases (A.2) are interesting. Following the similar arguments, we see nontrivial cases are exactly the same 5 cases as we summarized in §A.

Lemma B.2 *The inequality (2.23) holds for* $\alpha \leq \beta \leq \gamma$*, and the equality sign holds if and only if* $\beta = \alpha$ *or* $\beta = \gamma$*.*

This lemma actually implies that (2.23) holds if β lies between α and γ for all α, $\gamma \in \mathbb{R}$, not just these α and γ satisfying (A.1).
Proof: Assume $\alpha \neq \beta \neq \gamma$. Because of (A.1), we have $\gamma - \alpha = \gamma - \beta + \beta - \alpha$, and thus

$$\begin{aligned}
\widetilde{\text{RelDist}}(\alpha,\gamma) &= \frac{\gamma-\alpha}{\sqrt{|\alpha\gamma|}} = \frac{\gamma-\beta}{\sqrt{|\alpha\gamma|}} + \frac{\beta-\alpha}{\sqrt{|\alpha\gamma|}} \\
&= \frac{\gamma-\beta}{\sqrt{|\gamma\beta|}} + \frac{\beta-\alpha}{\sqrt{|\beta\alpha|}} \\
&\quad +(\gamma-\beta)\left(\frac{1}{\sqrt{|\alpha\gamma|}} - \frac{1}{\sqrt{|\gamma\beta|}}\right) + (\beta-\alpha)\left(\frac{1}{\sqrt{|\alpha\gamma|}} - \frac{1}{\sqrt{|\beta\alpha|}}\right) \\
&= \widetilde{\text{RelDist}}(\alpha,\beta) + \widetilde{\text{RelDist}}(\beta,\gamma) \\
&\quad +(\gamma-\beta)\frac{\sqrt{|\beta|}-\sqrt{|\alpha|}}{\sqrt{|\alpha\beta\gamma|}} - (\beta-\alpha)\frac{\sqrt{\gamma}-\sqrt{|\beta|}}{\sqrt{|\alpha\beta\gamma|}}.
\end{aligned}$$

Now if $\alpha < \beta \leq |\alpha| \leq \gamma$, then $\sqrt{|\beta|} - \sqrt{|\alpha|} \leq 0$ and $\sqrt{|\beta|} - \sqrt{\gamma} < 0$, and thus

$$(\gamma - \beta)\frac{\sqrt{|\beta|} - \sqrt{|\alpha|}}{\sqrt{|\alpha\beta\gamma|}} - (\beta - \alpha)\frac{\sqrt{\gamma} - \sqrt{|\beta|}}{\sqrt{|\alpha\beta\gamma|}} < 0.$$

Hence $\widetilde{\mathrm{RelDist}}(\alpha, \gamma) < \widetilde{\mathrm{RelDist}}(\alpha, \beta) + \widetilde{\mathrm{RelDist}}(\beta, \gamma)$. Consider now $|\alpha| < \beta < \gamma$. Then

$$\begin{aligned}
&(\gamma - \beta)\frac{\sqrt{|\beta|} - \sqrt{|\alpha|}}{\sqrt{|\alpha\beta\gamma|}} - (\beta - \alpha)\frac{\sqrt{|\gamma|} - \sqrt{|\beta|}}{\sqrt{|\alpha\beta\gamma|}} \\
\leq\ & (\gamma - \beta)\frac{\sqrt{\beta} - \sqrt{|\alpha|}}{\sqrt{|\alpha\beta\gamma|}} - (\beta - |\alpha|)\frac{\sqrt{\gamma} - \sqrt{\beta}}{\sqrt{|\alpha\beta\gamma|}} \\
=\ & -\frac{(\sqrt{\gamma} - \sqrt{\beta})(\sqrt{\beta} - \sqrt{|\alpha|})(\sqrt{\gamma} - \sqrt{|\alpha|})}{\sqrt{\alpha\beta\gamma}} \\
<\ & 0,
\end{aligned}$$

as required. ∎

Lemma B.3 *(2.23) holds for $\alpha\gamma \geq 0$.*

Proof: Lemma B.2 shows that (2.23) is true if $\alpha \leq \beta \leq \gamma$. If either $\beta < \alpha$ or $\gamma < \beta$, (2.23) follows from Property 6 of Proposition 2.8. ∎

As a immediate consequence of Lemma B.3, we have

Proposition B.1 $\widetilde{\mathrm{RelDist}}$ *is a metric on $\mathbb{R}_{\geq 0}$.*

Lemma B.4 *If $\alpha < 0 < -\alpha \leq \gamma \leq \beta$, then the inequality (2.23) holds, and the equality sign holds if and only if $\beta = \gamma$.*

Proof: Assume $\beta \neq \gamma$. By Lemma B.1, we may assume $\alpha = -1$. Then we want to have

$$\frac{\gamma + 1}{\sqrt{\gamma}} < \frac{\beta + 1}{\sqrt{\beta}} + \frac{\beta - \gamma}{\sqrt{\beta\gamma}},$$

or equivalently,

$$(\gamma + 1)\sqrt{\beta} - (\beta + 1)\sqrt{\gamma} + (\beta - \gamma) < 0.$$

Since

$$\begin{aligned}
(\gamma + 1)\sqrt{\beta} - (\beta + 1)\sqrt{\gamma} - (\beta - \gamma) &= \gamma\sqrt{\beta} + \sqrt{\beta} - \beta\sqrt{\gamma} - \sqrt{\gamma} + (\gamma - \beta) \\
&= \sqrt{\beta\gamma}(\sqrt{\gamma} - \sqrt{\beta}) + \sqrt{\beta} - \sqrt{\gamma} + (\gamma - \beta) \\
&= (\sqrt{\gamma} - \sqrt{\beta})(\sqrt{\beta\gamma} - 1 + \sqrt{\gamma} + \sqrt{\beta}) \\
&< 0,
\end{aligned}$$

as was to be shown. ∎

Lemma B.5 *When $\alpha = -1 < 0 < -\alpha \leq \gamma$, the inequality (2.23) holds for all $\beta \leq \alpha = -1$ if and only if $\gamma \leq 3 + 2\sqrt{2}$. If, however, $\gamma > 3 + 2\sqrt{2}$, then (2.23) holds for $\beta \leq -\sqrt{\gamma}\frac{\sqrt{\gamma}-1}{\sqrt{\gamma}+1}$.*

Proof: The inequality

$$\frac{\gamma+1}{\sqrt{\gamma}} \leq \frac{-1-\beta}{\sqrt{-\beta}} + \frac{\gamma-\beta}{\sqrt{-\gamma\beta}}$$

is equivalent to

$$(\gamma+1)\sqrt{-\beta} - (-1-\beta)\sqrt{\gamma} - (\gamma-\beta) \leq 0.$$

Write $-\beta = \xi^2$, so the above inequality reads

$$-\xi^2(\sqrt{\gamma}+1) + (\gamma+1)\xi + (\sqrt{\gamma}-\gamma) \leq 0. \tag{B.1}$$

So that (2.23) holds for all $\beta \leq \alpha = -1$ requires the inequality (B.1) is true for all $\xi \geq 1$. Since the two zeros of $-\xi^2(\sqrt{\gamma}+1) + (\gamma+1)\xi + (\sqrt{\gamma}-\gamma)$ are $\xi = 1$ and $\xi = \frac{\gamma-\sqrt{\gamma}}{\sqrt{\gamma}+1}$, and

$$\frac{\gamma-\sqrt{\gamma}}{\sqrt{\gamma}+1} \leq 1$$

gives $\gamma \leq 3 + 2\sqrt{2}$, we know that (2.23) holds for all $\beta \leq \alpha = -1$ if and only if $\gamma \leq 3 + 2\sqrt{2}$. If, however, $\gamma > 3 + 2\sqrt{2}$, then (2.23) is violated for $-\sqrt{\gamma}\frac{\sqrt{\gamma}-1}{\sqrt{\gamma}+1} < \beta < -1$. ∎

We may summarize how (2.23) is doing under the 5 distinguished cases.

1. (2.23) survives to *Case 1* by Lemma B.2;
2. (2.23) survives to *Case 2* by Lemma B.3;
3. (2.23) survives to *Case 3* by Lemma B.4;
4. (2.23) dies at *Cases 4* and/or *5*, unless $\gamma \leq 3 + 2\sqrt{2}$ by Lemma B.5.

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