The 48 Sets of Minimal Density MDS RAID-6 Matrices for a Word Size of Eight

James S. Plank

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University of Tennessee
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Please see the paper [Plank08] for all terminology related to RAID-6 coding using bit matrices.

There are 48 distinct sets of minimal density MDS RAID-6 codes with a word size $w=8$. Each may be defined by eight matrices, X_0, ..., X_7, where X_0 is always equal to an identity matrix. For example, Figure 1 shows the X_i matrices for the best matrix.

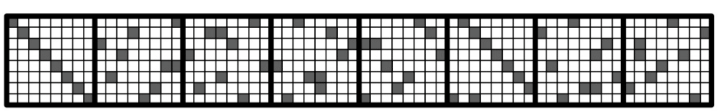

Figure 1: The eight X_i matrices for the best minimal density MDS matrix for $w=8$.

We represent each X_i ($i > 0$) with a permutation matrix and an extra one. The permutation matrix may be represented by a vector Π_i which has w integer elements $\pi_{i,0}$, ..., $\pi_{i,w-1}$. $\pi_{i,j}$ is the column which contains the location of the one in row j. In order to be a valid permutation matrix, Π_i must be such that $0 \le \pi_{ij} < w$ and if $j \ne j'$ then $\pi_{i,j} \ne \pi_{i,j'}$.

We can represent a matrix X_i with its permutation matrix Π_i plus a row and column identifying the extra one. We will use the following notation to represent X_i:

$$X_i = \{\Pi_i, r_i, c_i\}$$

For example, X_1 in Figure 1 may be represented as {(7,3,0,2,6,1,5,4),4,7}.

The following table enumerates the 48 sets of minimal density MDS matrices. In each matrix, X_1 is an identity matrix, and thus is not specified. The value f at the end of each row denotes the overhead factor of decoding with that matrix. This is the average overhead over optimal when decoding from two disk failures.

Note, the first row of the table is the matrix in Figure 1.

#	X_1	X_2	X_3	X_4	X_5	X_6	X_7	f
1	{(7,3,0,2,6,1,5,4),4,7}	{(6,2,4,0,7,3,1,5),1,3}	{(2,5,7,6,0,3,4,1),5,4}	{(5,6,1,7,2,4,3,0),2,0}	{(1,2,3,4,5,6,7,0),7,2}	{(3,0,6,5,1,7,4,2),6,5}	{(4,7,1,5,3,2,0,6),3,1}	1.18452
2	{(5,7,0,2,6,4,1,3),1,1}	{(6,3,1,4,7,2,5,0),3,2}	{(2,5,7,6,3,0,1,4),6,3}	{(4,7,6,0,1,2,3,5),5,7}	{(1,2,3,4,5,7,0,6),7,4}	{(3,4,5,7,0,1,2,6),2,6}	{(7,0,5,1,3,6,4,2),4,5}	1.18532
3	{(7,4,6,1,5,2,0,3),4,4}	{(1,2,3,4,7,0,5,6),7,3}	{(6,7,1,2,5,3,4,0),3,5}	{(3,4,0,5,2,7,1,6),1,6}	{(5,6,3,7,0,1,2,4),2,1}	{(2,3,4,0,6,1,7,5),5,7}	{(4,0,5,2,3,6,7,1),6,2}	1.18552
4	{(1,2,3,4,5,6,7,0),7,4}	{(3,6,0,1,7,2,4,5),5,6}	{(7,0,1,2,3,4,5,6),4,7}	{(6,7,5,0,2,3,1,4),2,1}	{(2,6,4,5,1,7,3,0),1,0}	{(5,3,7,4,6,2,0,1),3,2}	{(4,5,6,7,3,0,1,2),6,3}	1.18571
5	{(1,2,3,4,5,6,7,0),7,6}	{(2,5,4,1,7,0,3,6),3,4}	{(6,0,4,7,3,2,5,1),2,5}	{(4,6,7,2,1,3,5,0),6,0}	{(5,4,6,1,2,7,0,3),4,1}	{(3,7,0,5,2,6,1,4),5,2}	{(7,3,1,0,6,4,2,5),1,7}	1.18571
6	{(1,2,3,4,5,6,7,0),7,4}	{(2,5,6,0,3,7,1,4),6,3}	{(6,7,5,2,1,4,3,0),2,0}	{(3,6,0,4,7,1,2,5),3,6}	{(4,6,5,7,2,3,0,1),1,5}	{(7,4,1,6,0,2,5,3),5,7}	{(5,0,7,1,3,2,4,6),4,2}	1.18591
7	{(4,3,6,0,7,2,1,5),6,2}	{(7,0,5,2,1,4,3,6),2,6}	{(3,7,5,6,0,1,2,4),1,5}	{(6,5,7,1,3,2,4,0),5,3}	{(5,4,0,7,3,6,1,2),4,1}	{(1,2,3,4,5,7,0,6),7,4}	{(2,7,1,4,6,0,5,3),3,7}	1.18591
8	{(6,4,1,5,7,2,3,0),5,4}	{(7,6,5,1,2,3,0,4),2,1}	{(1,2,7,0,3,4,5,6),7,3}	{(4,0,5,7,3,6,1,2),4,5}	{(2,7,3,1,5,0,4,6),3,6}	{(5,4,0,2,6,1,7,3),1,7}	{(3,5,4,6,0,2,7,1),6,2}	1.18591
9	{(1,2,3,4,5,7,0,6),7,3}	{(7,0,6,1,2,4,3,5),3,4}	{(2,3,7,5,0,4,1,6),5,6}	{(5,7,0,1,6,3,2,4),6,1}	{(6,5,1,2,7,0,4,3),1,7}	{(3,4,5,6,7,1,2,0),4,2}	{(4,5,3,0,1,6,7,2),2,5}	1.18611
10	{(1,2,3,4,7,0,5,6),7,2}	{(2,6,5,0,3,7,4,1),6,5}	{(7,3,5,2,0,4,1,6),2,6}	{(6,0,7,1,2,3,4,5),5,4}	{(4,5,0,7,1,6,3,2),3,1}	{(5,7,4,6,1,3,2,0),4,3}	{(3,2,6,7,5,1,0,4),1,7}	1.18611
11	{(3,4,5,7,0,6,1,2),6,5}	{(4,7,1,0,6,2,5,3),1,2}	{(1,2,3,4,5,7,0,6),7,4}	{(7,0,5,2,1,4,3,6),2,6}	{(2,3,6,4,7,0,1,5),3,1}	{(6,5,7,1,3,2,4,0),5,3}	{(5,7,0,6,3,1,2,4),4,7}	1.18611
12	{(5,7,0,2,6,4,1,3),1,1}	{(7,0,6,4,1,2,3,5),3,2}	{(2,5,7,6,3,0,1,4),6,3}	{(3,7,5,1,0,6,4,2),2,7}	{(4,3,1,0,7,2,5,6),5,6}	{(1,2,3,4,5,7,0,6),7,4}	{(6,4,5,7,3,1,2,0),4,5}	1.18611
13	{(6,7,3,1,5,2,4,0),5,1}	{(7,4,5,2,6,1,0,3),2,4}	{(4,0,5,7,3,6,1,2),4,5}	{(3,5,4,6,0,2,7,1),6,2}	{(5,6,0,1,2,3,7,4),3,7}	{(1,2,7,0,3,4,5,6),7,3}	{(2,4,1,5,7,0,3,6),1,6}	1.18611
14	{(3,6,0,7,1,2,5,4),5,6}	{(1,2,3,4,5,6,7,0),7,4}	{(2,3,6,4,7,0,1,5),3,1}	{(6,7,4,5,0,2,1,3),6,2}	{(4,5,1,6,3,7,2,0),4,0}	{(5,6,7,2,3,4,0,1),1,3}	{(7,4,5,0,6,1,3,2),2,7}	1.18631
15	{(4,6,1,0,7,2,3,5),5,6}	{(1,2,3,4,5,6,7,0),7,4}	{(2,4,6,7,3,1,5,0),4,0}	{(6,7,4,5,0,2,1,3),6,2}	{(3,5,0,4,6,7,1,2),3,1}	{(5,6,7,2,3,4,0,1),1,3}	{(7,3,5,6,1,0,2,4),2,7}	1.18631
16	{(6,7,5,4,2,1,3,0),4,5}	{(4,3,6,2,1,0,7,5),1,7}	{(7,4,5,1,6,3,0,2),2,1}	{(3,0,1,5,2,6,7,4),6,2}	{(5,3,0,6,7,4,2,1),5,3}	{(1,2,7,0,3,4,5,6),7,4}	{(2,5,3,1,0,7,4,6),3,6}	1.18651
17	{(5,4,0,1,7,6,3,2),3,4}	{(2,5,4,1,6,0,7,3),6,1}	{(6,4,3,7,5,2,1,0),1,2}	{(3,7,1,5,0,2,4,6),5,6}	{(1,2,7,0,3,4,5,6),7,3}	{(7,6,5,2,3,1,0,4),4,5}	{(4,0,5,6,2,3,7,1),2,7}	1.18671
18	{(6,4,1,7,5,3,2,0),4,4}	{(4,0,5,2,3,6,7,1),6,2}	{(7,6,3,1,5,2,0,4),2,5}	{(3,4,0,5,2,7,1,6),1,6}	{(2,3,4,0,6,1,7,5),5,7}	{(5,7,6,2,0,1,4,3),3,1}	{(1,2,3,4,7,0,5,6),7,3}	1.18671
19	{(7,4,5,6,2,3,0,1),4,5}	{(3,0,1,5,2,6,7,4),6,2}	{(6,3,5,4,7,1,2,0),2,3}	{(4,3,6,2,1,0,7,5),1,7}	{(5,7,0,1,6,4,3,2),5,1}	{(2,5,3,1,0,7,4,6),3,6}	{(1,2,7,0,3,4,5,6),7,4}	1.18730
20	{(1,2,3,4,5,6,7,0),7,1}	{(5,3,0,6,7,4,2,1),5,6}	{(6,7,5,1,2,0,3,4),6,2}	{(4,6,7,5,2,1,0,3),4,7}	{(2,5,4,0,1,7,3,6),1,3}	{(7,5,1,6,0,3,4,2),3,5}	{(3,0,7,2,6,4,1,5),2,4}	1.18750
21	{(1,2,3,4,5,6,7,0),7,1}	{(6,4,5,0,3,7,2,1),2,3}	{(3,7,0,6,1,2,4,5),6,2}	{(2,6,7,5,0,1,4,3),1,4}	{(7,6,4,1,3,0,5,2),4,6}	{(5,3,1,7,6,2,0,4),5,7}	{(4,0,5,7,2,3,1,6),3,5}	1.18750
22	{(1,2,3,4,5,7,0,6),7,3}	{(7,6,1,5,0,2,4,3),6,5}	{(2,4,7,5,3,0,1,6),3,6}	{(3,0,5,6,7,1,4,2),5,4}	{(5,7,6,1,2,4,3,0),1,2}	{(6,3,0,7,2,1,5,4),4,1}	{(4,7,3,0,1,6,2,5),2,7}	1.18750
23	{(1,2,3,4,7,0,5,6),7,2}	{(5,6,0,2,3,7,4,1),4,7}	{(4,2,6,7,3,1,0,5),1,3}	{(6,4,5,0,2,7,1,3),5,1}	{(3,0,4,5,1,6,7,2),2,5}	{(2,7,4,6,5,3,1,0),6,4}	{(7,3,1,5,0,4,2,6),3,6}	1.18750
24	{(5,4,0,1,7,6,3,2),3,4}	{(2,5,4,1,6,0,7,3),6,1}	{(4,0,3,7,5,2,1,6),5,6}	{(7,4,5,6,2,3,0,1),1,5}	{(6,7,1,5,3,2,4,0),4,2}	{(1,2,7,0,3,4,5,6),7,3}	{(3,6,5,2,0,1,7,4),2,7}	1.18790
25	{(1,2,3,4,5,6,7,0),7,4}	{(2,3,5,0,6,7,1,4),6,5}	{(3,6,0,1,7,2,4,5),5,6}	{(5,0,7,4,2,3,1,6),3,1}	{(6,7,5,2,1,4,3,0),2,0}	{(7,6,4,5,0,1,2,3),1,7}	{(4,5,6,7,3,2,0,1),4,2}	1.18869
26	{(1,2,3,4,5,6,7,0),7,4}	{(2,3,5,0,6,7,1,4),6,5}	{(7,0,1,2,3,4,5,6),4,7}	{(3,6,4,5,7,2,0,1),1,2}	{(6,7,5,1,3,0,4,2),2,3}	{(4,5,6,7,1,2,3,0),5,0}	{(5,6,7,4,0,1,2,3),3,6}	1.18869
27	{(1,2,3,4,5,6,7,0),7,4}	{(2,5,6,0,3,7,1,4),6,3}	{(4,6,0,7,1,2,3,5),1,2}	{(6,7,1,2,3,4,5,0),4,0}	{(5,6,7,4,0,1,2,3),3,6}	{(3,4,5,6,7,0,1,2),2,1}	{(7,3,4,5,6,2,0,1),5,7}	1.18869
28	{(1,2,3,4,5,6,7,0),7,4}	{(5,3,7,0,6,2,1,4),5,1}	{(4,0,5,7,3,1,2,6),2,3}	{(3,4,5,6,7,0,1,2),6,5}	{(6,7,1,4,0,2,5,3),3,2}	{(7,5,6,2,3,4,0,1),4,7}	{(2,6,4,5,1,7,3,0),1,0}	1.18869
29	{(1,2,3,4,5,6,7,0),7,4}	{(3,0,5,2,7,4,1,6),2,1}	{(2,3,5,4,6,7,0,1),3,5}	{(5,4,7,6,2,3,1,0),6,0}	{(4,6,1,7,3,0,5,2),4,6}	{(7,6,4,5,0,1,2,3),1,7}	{(6,7,0,1,3,2,4,5),5,3}	1.18909
48	{(1,2,3,4,5,6,7,0),7,4}	{(3,0,5,2,7,4,1,6),2,1}	{(2,3,4,5,6,7,1,0),6,0}	{(6,7,5,4,1,0,3,2),3,5}	{(5,6,7,0,3,1,2,4),1,3}	{(4,5,6,7,3,2,0,1),4,2}	{(7,4,1,6,0,2,5,3),5,7}	1.18930
31	{(1,2,3,4,5,6,7,0),7,7}	{(7,4,1,2,6,0,5,3),3,4}	{(3,4,6,1,7,2,0,5),1,6}	{(2,7,6,5,3,1,4,0),2,0}	{(4,5,0,2,1,7,3,6),4,2}	{(6,3,5,0,1,4,7,2),6,1}	{(5,6,4,7,0,3,2,1),5,5}	1.19028
32	{(3,4,0,6,2,7,5,1),3,3}	{(6,7,5,2,0,4,1,3),1,4}	{(4,6,1,7,5,3,0,2),4,1}	{(7,3,1,5,6,0,2,4),2,2}	{(1,2,3,4,5,6,7,0),7,5}	{(2,5,6,1,7,4,3,0),5,0}	{(5,7,4,0,3,1,2,6),6,7}	1.19028
33	{(1,2,3,4,5,6,7,0),7,4}	{(6,7,5,0,2,3,1,4),6,5}	{(3,6,4,5,7,2,0,1),5,6}	{(2,5,6,4,0,7,1,3),3,1}	{(5,0,7,1,3,2,4,6),4,2}	{(7,6,0,2,1,4,3,5),1,7}	{(4,3,5,7,6,1,2,0),2,0}	1.19087
34	{(1,2,3,4,5,6,7,0),7,4}	{(6,7,5,0,2,3,1,4),6,5}	{(3,6,4,5,7,2,0,1),1,2}	{(5,4,7,6,0,2,1,3),5,1}	{(4,6,5,7,1,0,3,2),2,6}	{(2,5,6,1,3,7,4,0),4,0}	{(7,3,0,4,6,1,2,5),3,7}	1.19127
35	{(1,2,3,4,5,6,7,0),7,4}	{(3,6,4,5,7,2,0,1),5,6}	{(6,7,5,0,2,3,1,4),2,1}	{(2,6,5,1,0,7,4,3),1,5}	{(4,3,0,7,6,2,1,5),6,2}	{(5,4,7,6,3,1,2,0),4,0}	{(7,5,6,4,1,0,3,2),3,7}	1.19186
36	{(1,2,3,4,5,6,7,0),7,4}	{(6,7,5,1,3,0,4,2),2,3}	{(2,3,5,4,6,7,0,1),3,5}	{(3,5,6,2,7,4,1,0),6,0}	{(7,0,4,5,3,1,2,6),4,7}	{(5,4,7,6,0,2,1,3),5,1}	{(4,6,0,7,1,2,3,5),1,2}	1.19306
37	{(1,2,3,4,5,6,7,0),7,4}	{(2,6,5,0,1,7,3,4),2,6}	{(3,5,6,1,7,2,4,0),5,0}	{(6,7,4,5,3,0,1,2),4,1}	{(5,6,7,2,3,4,0,1),1,3}	{(4,3,0,7,6,2,1,5),6,2}	{(7,0,1,4,2,3,5,6),3,7}	1.19325
38	{(1,2,3,4,5,6,7,0),7,4}	{(5,3,7,0,6,2,1,4),5,1}	{(3,6,0,4,7,1,2,5),3,6}	{(7,4,5,6,2,3,0,1),2,7}	{(2,6,5,1,0,7,4,3),1,5}	{(4,5,6,7,3,0,1,2),6,3}	{(6,7,1,2,3,4,5,0),4,0}	1.19325
39	{(1,2,3,4,5,6,7,0),7,4}	{(5,6,7,1,0,2,4,3),1,2}	{(6,7,1,0,3,2,5,4),5,3}	{(2,3,4,5,6,7,1,0),6,0}	{(7,5,6,2,3,4,0,1),4,7}	{(3,0,5,4,7,1,2,6),3,5}	{(4,6,5,7,1,0,3,2),2,6}	1.19425
40	{(1,2,3,4,5,6,7,0),7,4}	{(2,6,5,0,1,7,3,4),2,6}	{(4,6,1,7,2,3,5,0),1,0}	{(3,4,5,6,7,0,1,2),6,5}	{(7,0,4,5,3,1,2,6),4,7}	{(6,7,0,1,3,2,4,5),5,3}	{(5,3,7,4,6,2,0,1),3,2}	1.19464
41	{(1,2,3,4,5,6,7,0),7,7}	{(4,5,7,2,6,0,3,1),6,4}	{(5,0,1,7,2,3,4,6),3,2}	{(2,3,4,6,7,1,0,5),2,1}	{(7,4,6,0,3,1,5,2),5,0}	{(6,7,0,5,2,4,1,3),4,3}	{(3,6,5,1,0,7,2,4),1,5}	1.19504
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43	{(1,2,3,4,5,6,7,0),7,4}	{(5,6,7,1,0,2,4,3),1,2}	{(4,5,6,7,1,2,3,0),5,0}	{(7,0,1,4,2,3,5,6),3,7}	{(3,4,5,6,7,0,1,2),2,1}	{(6,7,0,2,3,4,1,5),6,3}	{(2,6,4,5,3,7,0,1),4,6}	1.19544
44	{(6,5,0,7,1,3,4,2),6,0}	{(5,4,7,6,3,2,0,1),5,3}	{(4,7,6,1,2,0,3,5),3,5}	{(7,6,4,5,0,1,2,3),1,7}	{(2,6,1,0,3,7,5,4),4,6}	{(3,0,5,2,7,4,1,6),2,1}	{(1,2,3,4,5,6,7,0),7,4}	1.19544
45	{(1,2,3,4,5,6,7,0),7,4}	{(3,5,6,1,7,2,4,0),5,0}	{(5,0,7,4,2,3,1,6),3,1}	{(6,7,4,5,0,2,1,3),6,2}	{(4,6,1,7,3,0,5,2),4,6}	{(2,4,5,6,3,7,0,1),2,3}	{(7,6,0,2,1,4,3,5),1,7}	1.19583
46	{(1,2,3,4,5,6,7,0),7,6}	{(3,0,5,7,1,4,2,6),2,3}	{(5,4,6,1,2,7,0,3),3,2}	{(6,7,1,2,3,0,4,5),1,4}	{(7,3,5,6,0,1,4,2),6,5}	{(2,5,7,0,6,3,1,4),4,0}	{(4,6,0,5,7,2,3,1),5,1}	1.19603
47	{(1,2,3,4,5,6,7,0),7,7}	{(3,7,4,5,6,1,0,2),5,4}	{(7,6,0,1,2,4,5,3),4,3}	{(6,3,1,7,2,0,4,5),3,2}	{(2,5,6,0,3,7,1,4),2,0}	{(4,0,5,2,7,1,3,6),6,1}	{(5,4,7,6,0,3,2,1),1,5}	1.19603
48	{(6,0,7,1,3,2,5,4),4,1}	{(3,5,4,6,1,7,2,0),6,0}	{(4,3,6,5,7,0,1,2),3,6}	{(5,4,6,7,2,3,0,1),2,3}	{(1,2,3,4,5,6,7,0),7,5}	{(7,6,5,2,0,1,4,3),5,4}	{(2,7,1,0,6,4,3,5),1,2}	1.19623

Reference

[Plank 08] James S. Plank, ``The RAID-6 Liberation Codes,'' *FAST-2008: 6th Usenix Conference on File and Storage Technologies*, San Jose, CA, February, 2008.